A Joint Theory of Belief and Probability

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One reason why qualitative belief is so valuable is that it occupies a *more elementary* scale of measurement than quantitative belief.

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Two different paths lead to one and the same answer:

1. "$\leftarrow$" of the Lockean Thesis and the Logic of Absolute Belief
2. "$\rightarrow$" of the Lockean Thesis and the Logic of Conditional Belief

cf. Skyrms (1977), (1980) on resiliency.

Snow (1998), Dubois et al. (1998) on big-stepped probabilities.

An answer is crucial, for how else can we reconcile *traditional* philosophy of science, epistemology, philosophy of language, and cognitive science with:

"$\leftarrow$" of the Lockean Thesis and the Logic of Absolute Belief

Let W be a set of possible worlds, and let $\mathfrak{A}$ be an algebra of subsets of W (propositions) in which an agent is interested at a time.

We assume that $\mathfrak{A}$ is closed under countable unions (σ-algebra).

Let P be an agent's degree-of-belief function at the time.

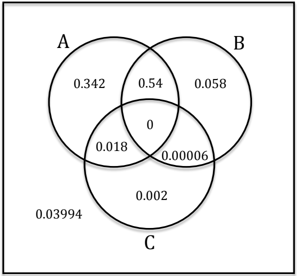
P1 (Probability) $P : \mathfrak{A} \to [0,1]$ is a probability measure on $\mathfrak{A}$.

$P(Y|X) = \frac{P(Y \cap X)}{P(X)}$, when $P(X) > 0$.

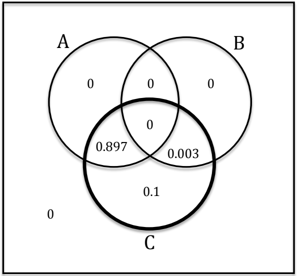
P2 (Countable Additivity) If $X_1, X_2, \ldots, X_n, \ldots$ are pairwise disjoint members of $\mathfrak{A}$, then

$$P(\bigcup_{n \in \mathbb{N}} X_n) = \sum_{n=1}^{\infty} P(X_n).$$

E.g., a probability measure P:



P conditionalized on C:



Accordingly, let *Bel* express an agent's beliefs.

- B1 (Logical Truth) $Bel(W)$.
- B2 (One Premise Logical Closure) For all $Y, Z \in \mathfrak{A}$:
  If $Bel(Y)$ and $Y \subseteq Z$, then $Bel(Z)$.
- B3 (Finite Conjunction) For all $Y, Z \in \mathfrak{A}$:
  If $Bel(Y)$ and $Bel(Z)$, then $Bel(Y \cap Z)$.
- B4 (General Conjunction) For $\mathcal{Y} = \{Y \in \mathfrak{A} \mid Bel(Y)\}$, $\bigcap \mathcal{Y}$ is a member of $\mathfrak{A}$, and $Bel(\bigcap \mathcal{Y})$.

It follows: There is a *strongest proposition* B_W, such that $Bel(Y)$ iff $Y \supseteq B_W$.

In order to spell out under what conditions these postulates are consistent with the "$\leftarrow$" of the Lockean thesis,

- $\mathrm{LT}_{\leftarrow}^{\geq r > \frac{1}{2}}$: $\quad Bel(X)$ if $P(X) \geq r > \frac{1}{2}$

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Definition

(P-Stability) For all $X \in \mathfrak{A}$:

X is P-stable^r iff for all $Y \in \mathfrak{A}$ with $Y \cap X \neq \varnothing$ and $P(Y) > 0$: $P(X|Y) > r$.

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So P-stable^r propositions have stably high probabilities under salient suppositions. (Examples: All X with $P(X) = 1$; $X = \varnothing$; and *many* more!)

Remark: If X is P-stable^r with $r \in \left[\frac{1}{2}, 1\right)$, then X is P-stable$^{\frac{1}{2}}$.

(cf. Skyrms 1977, 1980 on resiliency.)

Then the following representation theorem can be shown:

Theorem

Let Bel be a class of members of a σ-algebra $\mathfrak{A}$, and let $P : \mathfrak{A} \to [0,1]$. Then the following two statements are equivalent:

I. *P and Bel satisfy* P1, B1–B4, *and* $\mathrm{LT}_{\leftarrow}^{\geq P(B_W) > \frac{1}{2}}$.

II. *P satisfies* P1 *and there is a (uniquely determined) $X \in \mathfrak{A}$, such that*

– *X is a non-empty P-stable$^{\frac{1}{2}}$ proposition,*

– *if $P(X) = 1$ then X is the least member of $\mathfrak{A}$ with probability 1; and:*

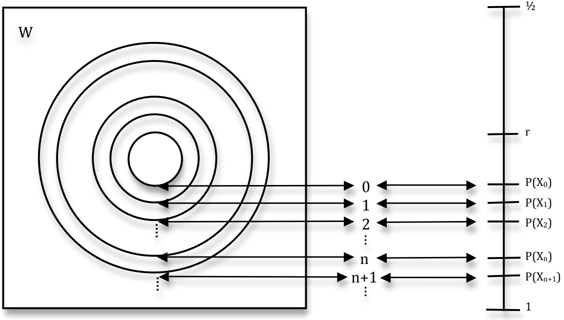
For all $Y \in \mathfrak{A}$:

$$Bel(Y) \text{ if and only if } Y \supseteq X$$

(and hence, $B_W = X$).

And either side implies the *full* $\mathrm{LT}_{\leftrightarrow}^{\geq P(B_W) > \frac{1}{2}}$: $Bel(X)$ iff $P(X) \geq P(B_W) > \frac{1}{2}$.

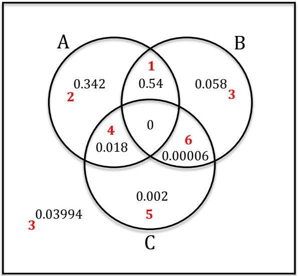
With P2 one can prove: The class of P-stable^r propositions X in $\mathfrak{A}$ with $P(X) < 1$ is *well-ordered* with respect to the subset relation.



This implies: If there is a non-empty P-stable^r X in $\mathfrak{A}$ with $P(X) < 1$ at all, then there is also a *least* such X.

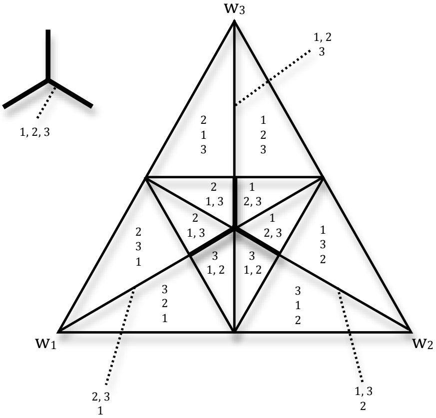
Example: Let P be as in the initial example.

6. $P(\{w_7\}) = 0.00006$ ("Ranks")
5. $P(\{w_6\}) = 0.002$
4. $P(\{w_5\}) = 0.018$
3. $P(\{w_3\}) = 0.058$, $P(\{w_4\}) = 0.03994$
2. $P(\{w_2\}) = 0.342$
1. $P(\{w_1\}) = 0.54$

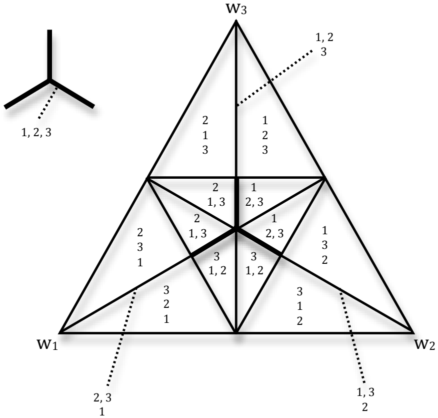


This yields the following P-stable$^{\frac{1}{2}}$ sets:

- $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ (≥ 1.0)
- $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ (≥ 0.99994)
- $\{w_1, w_2, w_3, w_4, w_5\}$ (≥ 0.99794)
- $\{w_1, w_2, w_3, w_4\}$ (≥ 0.97994)
- $\{w_1, w_2\}$ (≥ 0.882)
- $\{w_1\}$ (≥ 0.54) ("Spheres")



Almost all P here have a least P-stable$^{\frac{1}{2}}$ set X with $P(X) < 1$!



Almost all P here have a least *P*-stable$^{\frac{1}{2}}$ set X with $P(X) < 1$!

Hence, for *lots of P* there is an r, such that there is a *Bel* with:

- B1–4 Logical closure of *Bel*.
- $LT^{>r}_{\leftrightarrow}$ For all X: $Bel(X)$ iff $P(X) > r$.
- NT There is an X, such that $Bel(X)$ and $P(X) < 1$.

But occasionally there is *no* X, such that $Bel(X)$ and $P(X) < 1$:

- *Lottery Paradox:* Given a uniform measure P on a finite set W of worlds, W is the only P-stable^r set with $r \geq \frac{1}{2}$; so only W is to be believed then.

  This makes good sense: the agent's degrees of belief don't give her much of a hint of what to believe. *That is why the agent ought to be cautious.*

Moral:

- Given P and a cautiousness threshold r, the agent's *Bel* is determined uniquely by the Lockean thesis.
- *Bel* is even closed logically iff

  Bel is given by a P-stable$^{\frac{1}{2}}$ set X with $P(X) = r > \frac{1}{2}$.
- So the Lockean thesis and the logical closure of belief are jointly satisfiable as long as the threshold r is *co-determined* by P.
- From the probabilistic point of view, belief *simpliciter* corresponds to *resiliently* high probability—which seem plausible even on independent grounds.

"$\rightarrow$" of the Lockean Thesis and Conditional Belief

Now let '*Bel*' express an agent's *conditional* beliefs:

$Bel(Y|X)$ iff the agent has *a belief in* Y *on the supposition of* X.
$Bel(Y)$ iff $Bel(Y|W)$ iff the agent *believes* Y *(unconditionally)*.

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In this way, we can reformulate the axioms of belief expansion/revision; e.g.,

- (Finite Conjunction) If $\neg Bel(\neg X|W)$, then for all $Y, Z \in \mathfrak{A}$:
  If $Bel(Y|X)$ and $Bel(Z|X)$, then $Bel(Y \cap Z|X)$.

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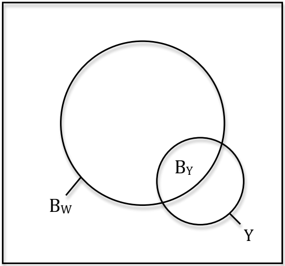
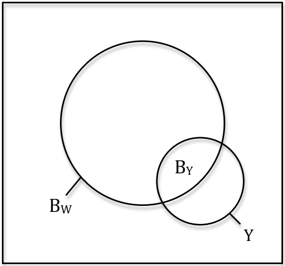
  If $Bel(Y|X)$ and $Bel(Z|X)$, then $Bel(Y \cap Z|X)$.

From this (and more) we have again: For every $X \in \mathfrak{A}$ [with $\neg Bel(\neg X|W)$], there is a *strongest proposition* B_X, such that $Bel(Y|X)$ iff $Y \supseteq B_X$.



- (Expansion) For all $Y \in \mathfrak{A}$ such that $Y \cap B_W \neq \emptyset$: $B_Y = Y \cap B_W$.

This "quasi-Bayesian" postulate is contained in the classic qualitative theory of belief revision (AGM 1985, Gärdenfors 1988).



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Indeed, the full AGM theory includes the stronger postulate

- (Revision) For all $X, Y \in \mathfrak{A}$ such that $Y \cap B_X \neq \emptyset$: $B_{X \cap Y} = Y \cap B_X$

which entails that *Bel* is given by a total pre-order (sphere system) of worlds.

We get the following representation theorem for belief expansion and "$\rightarrow$" of the Lockean Thesis (with r *independent* of P):

Theorem

The following two statements are equivalent:

I. *P and Bel satisfy* P1*, the AGM axioms for belief expansion, and* $\mathrm{LT}^{\geq r}_{\rightarrow}$.

II. *P satisfies* P1*, and there is a (uniquely determined) $X \in \mathfrak{A}$, such that X is a non-empty P-stable^r proposition, and $Bel(\cdot|\cdot)$ is given by X ($= B_W$).*

$\mathrm{LT}^{\geq r}_{\rightarrow}$ ("$\rightarrow$" of Lockean thesis) For all $Y \in \mathfrak{A}$, s.t. $P(Y) > 0$ and $Y \cap B_W \neq \varnothing$:
For all $Z \in \mathfrak{A}$, if $Bel(Z|Y)$, then $P(Z|Y) > r$.

And either side implies the *full* $\mathrm{LT}^{\geq P_Y(B_Y)}_{\leftrightarrow}$: $Bel(Z|Y)$ iff $P_Y(Z) \geq P_Y(B_Y) > r$.

And we have the following representation theorem for belief revision and "$\rightarrow$" of the Lockean Thesis (with r *independent* of P):

Theorem

The following two statements are equivalent:

I. *P and Bel satisfy* P1–P2, *the AGM axioms for belief revision, and* $\mathrm{LT}^{\geq r}_{\rightarrow}$.

II. *P satisfies* P1–P2, *and there is a (uniquely determined) chain $\mathcal{X}$ of non-empty P-stable^r propositions in $\mathfrak{A}$, such that $Bel(\cdot|\cdot)$ is given by $\mathcal{X}$ in a Lewisian sphere-system-like manner.*

$\mathrm{LT}^{\geq r}_{\rightarrow}$ ("$\rightarrow$" of Lockean thesis) For all $Y \in \mathfrak{A}$, s.t. $P(Y) > 0$:

For all $Z \in \mathfrak{A}$, if $Bel(Z|Y)$, then $P(Z|Y) > r$.

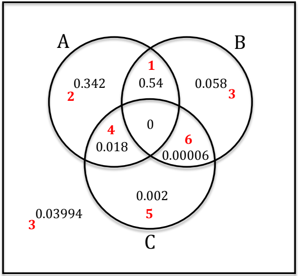
And either side implies the *full* $\mathrm{LT}^{\geq P_Y(B_Y)}_{\leftrightarrow}$: $Bel(Z|Y)$ iff $P_Y(Z) \geq P_Y(B_Y) > r$.

Example: Let P be again as in the example before.

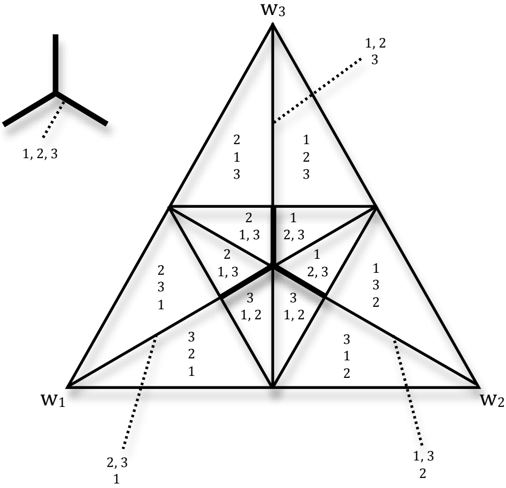
Then if $Bel(\cdot|\cdot)$ satisfies AGM, and if P and $Bel(\cdot|\cdot)$ jointly satisfy $LT^{>\frac{1}{2}}_{\rightarrow}$, then $Bel(\cdot|\cdot)$ must be given by some coarse-graining of the ranking in red below.

Choosing the maximal (most fine-grained) $Bel(\cdot|\cdot)$ yields the following:

- $Bel(A \wedge B | A)$ $(A \rightarrow A \wedge B)$
- $Bel(A \wedge B | B)$ $(B \rightarrow A \wedge B)$
- $Bel(A \wedge B | A \vee B)$ $(A \vee B \rightarrow A \wedge B)$
- $Bel(A | C)$ $(C \rightarrow A)$
- $\neg Bel(B | C)$ $(C \not\rightarrow B)$
- $Bel(A | C \wedge \neg B)$ $(C \wedge \neg B \rightarrow A)$
- $\neg Bel(B | \neg A)$ $(\neg A \not\rightarrow B)$



For three worlds again (and $r = \frac{1}{2}$), the maximal $Bel(\cdot|\cdot)$ as being determined by P and r are given by these rankings:



Moral:

- Given P and a threshold r, the agent's $Bel(\cdot|\cdot)$ is not determined uniquely by the "$\rightarrow$" of the Lockean thesis.
- But any such $Bel(\cdot|\cdot)$ is closed logically iff it is given by a sphere system of P-stable^r sets.
- Given P and a threshold r, the agent's *maximal* $Bel(\cdot|\cdot)$ amongst those that satisfy all of our postulates is determined uniquely.

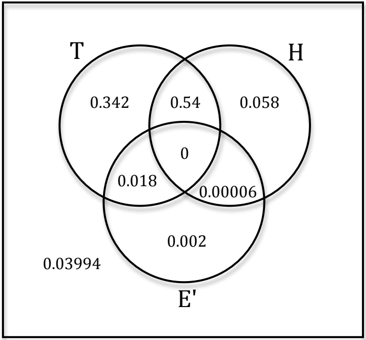
  (And there is always such a unique maximal choice Bel_P^r given a rather weak auxiliary assumption.)

As promised, we end up with a unified theory of belief and probability.

The theory is robust—two plausible paths lead to it.

Postscript

Our example P derives from *Bayesian Philosophy of Science* (Dorling 1979)



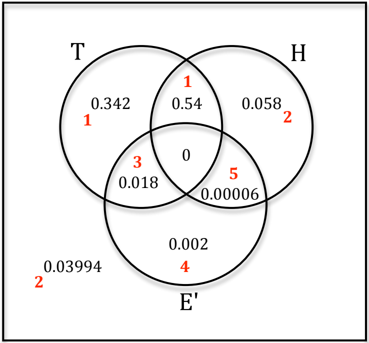
E': Observational result for the secular acceleration of the moon.

T: Relevant part of Newtonian mechanics.

H: Auxiliary hypothesis that tidal friction is negligible.

$P(T|E') = 0.8976$, $P(H|E') = 0.003$.

while I will insert definite numbers so as to simplify the mathematical working, nothing in my final qualitative interpretation... will depend on the precise numbers...



$$Bel_P^r(T|E'),\ Bel_P^r(\neg H|E')\ \text{(with } r = \tfrac{3}{4}\text{)}.$$

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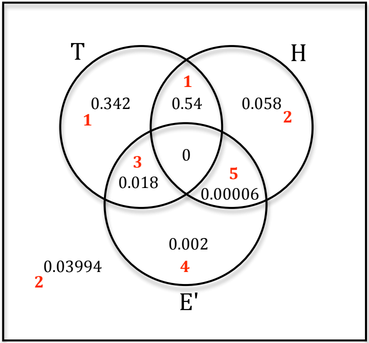

$$Bel_P^r(T|E'), \; Bel_P^r(\neg H|E') \;\; (\text{with } r = \tfrac{3}{4}).$$

... scientists always conducted their serious scientific debates in terms of finite qualitative subjective probability assignments to scientific hypotheses (Dorling 1979).