A Philosopher's Guide to Forcing: What is a generic set?

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Why?

- The **continuum hypothesis** (CH) poses a problem for our attempts to **grapple with the infinite**.
- Cohen showed us that CH was **independent** of ZFC.
- So either the question is **under-determined** or we ought to add some **more axioms**.
- Cohen showed this by adding a **generic set** using the technique of **forcing**.

- Forcing has been around for around **50 years**, but a certain enigma surrounds it.

  There are certainly moments in any mathematical discovery when the resolution of a problem takes place at such a subconscious level that, in retrospect, it ***seems impossible*** *to* ***dissect*** *it and* ***explain*** *its origin. Rather, the entire idea presents itself at once, often perhaps in a vague form, but gradually becomes more precise. [?]*

- Compare this situation to a **completeness proof**.
- There we want to show that **every consistent set is satisfiable**.
- So we take a consistent **set of sentences** and use that to **construct a model**.
- SIMPLE!

- The goal of this paper is to do **something similar**.
- I want to give an explanation of how forcing arguments work at, for want of a better term, a **conceptual level**.

Technical bits & pieces

- Let us start with an arbitrary first order language $\mathscr{L} = \{P, R, ..., f ...\}$ and some model $\mathscr{M}$ of it.
- Let $\mathscr{M} = \langle M, v\rangle$ be an arbitrary model of $\mathscr{L}$.
  - M is the domain.
  - $v = \{P^{\mathscr{M}}, R^{\mathscr{M}}, ..., f^{\mathscr{M}}, ...\}$ is an interpretation of the language.

- We add a **new 1-place relation** symbol $\dot{G}$ to the language and call the resultant expansion $\mathscr{L}_G$.
- Let G be an arbitrary subset of M and let this be the interpretation of $\dot{G}$ in the expansion of $\mathscr{M}$ for $\mathscr{L}_G$ which we **denote** $\mathscr{M}[G]$. This called a **generic extension**.
- The **game of forcing** is all about using $\dot{G}$ to represent an ***interesting*** object in models of the expanded language $\mathscr{L}_G$. We shall denote these models $\mathscr{M}[G]$.
- When everything is working in concert, we shall call such an object a ***generic element***.

Leading by examples

We now look at three **famous examples** of forcing arguments:

1. Cohen's proof that $V = L$ is not implied by ZFC;
2. Cohen's proof that CH is not implied by ZFC; and
3. Addison's proof that the arithmetically definable reals are not themselves arithmetically definable.

1. Constructible sets

- L is known as the ***constructible hierarchy*** and was developed by Gï¿œdel.

- $V = L$ is, loosely speaking, the statement that everything in the universe is ***constructible***.
- I'll provide a quick semi-formal definition of L and try to describe why it is interesting. L is **constructed in stages**.
- We **start** with the **empty set** and then at **every stage** we add **all the subsets** of the previous stage **that could be defined using elements of the previous stage** as parameters:
- $L_{\alpha+1} := \{x \subseteq L_\alpha \mid$ such that x is definable from a formula $\varphi(x, a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in L_\alpha\}$.
- We then **iterate this process over the entirety of the ordinals** and the result is called L.

- A couple of obvious reason for our interest in L are:

Fact

L is a model of ZFC: all of the axioms of ZFC are true there (assuming ZFC is consistent).

Fact

*L is defined using an **extremely simple** process. As such any model of set theory (of a certain very natural kind) must contain a copy of L and any two models will agree on what should be in L. It is the **thinnest model** of set theory.*

- However, this doesn't tell us that there are any other models $\mathscr{N}$ of ZFC;
  - which **properly contain** L; and
  - which contain elements which are **not constructible**;
  - i.e., $\mathscr{N} \models V \neq L$.

- Cohen addressed this question using forcing.

Theorem
ZFC $\nvdash V = L$.

- The strategy for proving this theorem is to ***find*** a model $\mathscr{N}$ in which it is **not the case** that $V = L$. We want to **make a model** which contains an element which is ***not constructible***.
- Our approach for doing this will be via a ***transformation***. We want to take an arbitrary model $\mathscr{M}$ of *ZFC* and **adjoin** to it a new element G in such a way that the resultant model $\mathscr{M}[G]$ is still a model of *ZFC*.

- We're going to let G be something that $\mathscr{M}$ ***would think*** is a set of natural numbers, but which is not in $\mathscr{M}$. We'll call G a **Cohen real**.
- So now we have two models of ZFC, $\mathscr{M}$ and $\mathscr{M}[G]$ such that $G \notin \mathscr{M}$.
- However, by Fact 2, we know that each of models must contain a copy of L, since it L so simple.
- But then clearly at most one of $\mathscr{M}$ and $\mathscr{M}[G]$ could be L: i.e., $\mathscr{M}$. Thus $\mathscr{M}[G]$ is a model of $V \neq L$.

- In a nutshell, we've simply **added a nonconstructible** element to the model.
- The **generic element** G did this for us. It was the **non-constructible** element.
- *Strictly speaking, there is a problem with this strategy in that there is more to say about what kind of models can be used here. However, we'll leave that in a black box for the moment.*

2. The Continuum Hypothesis

- Let's start with the famous example of the **continuum hypothesis** (CH).
- This says that:
  - there are **no cardinals between** ω and 2^ω; or
  - perhaps more naturally, **every set of natural numbers is either countable or has the size of the continuum**; or
  - every subset of $\mathscr{P}(\omega)$ can be put into bijective correspondence with a ω or $\mathscr{P}(\omega)$.

- We want to show:

Theorem
$ZFC \nvdash CH$

- Our strategy for proving this theorem is, again, to **find a model** $\mathscr{N}$ in which **it is not the case that** CH .

- Again, our approach for doing this will be via a **transformation**. We want to take an arbitrary (*countable*) model$\mathscr{M}$ of ZFC and add to it a new element G in such a way that the resultant model $\mathscr{M}[G]$ is still a model of ZFC .

- This time we are going to let G be a set of what $\mathscr{M}$ ***would think*** are ω_2 many distinct real numbers.
- We shall assume that we can show that $\mathscr{M}[G]$ is **still a model** of ZFC . Moreover, we can show that what $\mathscr{M}$ thought was ω_2, is still thought to be ω_2 by $\mathscr{M}[G]$.
- But **now we have a model** $\mathscr{M}[G]$ **in which there are** ω_2 **many real numbers**. Thus we have a model in which the CH is false and by soundness $ZFC + \neg CH$ is consistent.

- In a nutshell, we added ω_2 many real numbers to $\mathscr{M}$.
- Moreover G was that **collection of real numbers**.

3. Addison's theorem

- For this example, we turn to the **arithmetic** instead of **set theory.**
- We'll denote the **standard model** of arithmetic by $\mathbb{N}$.
- It's **not a perfect fit** with the other examples, but serves to **illustrate** that forcing is not merely a set theoretic technique.

Theorem

The collection of arithmetically definable classes is not itself arithmetically definable.

- A set of natural numbers $A \in \omega$ is **arithmetically definable** if there is some formula φ of the language of arithmetic such that:

$$n \in A \ \Leftrightarrow\ \mathbb{N} \models \varphi(n).$$

- A family of sets of naturals $\mathscr{B} \in \mathscr{P}(\omega)$ is **arithmetically definable** if there is some formula $\varphi(X)$ in the expanded language $\mathscr{L}_G$ such that:

$$G \in \mathscr{B} \ \Leftrightarrow\ \mathbb{N}[G] \models \varphi(\dot{G}).$$

- Observe that there are **only countably many** arithmetically definable families of sets of naturals, as there are **only countably many formulae** to define them.

- We aren't proving an **independence result** this time, so our **strategy** is slightly **different**.
- We are going to consider a **collection** of **real** numbers $\mathscr{G}$ each of which is a generic object G. This collection has some nice properties:
  - Given any **finite chunk** of information, p, about G, there are $2^{\aleph_0}$ many members of $\mathscr{G}$ which **could contain** that information.
  - For any particular formula $\varphi(X)$ of $\mathscr{L}_G$ there is a finite chunk of information p about G, which would **force** it to be the case that $\mathscr{M}[G] \models \varphi(\dot{G})$ iff that were true.

- We then prove the theorem by ***reductio***. Let $\mathscr{A}$ be the arithmetically definable families of sets of naturals.

- We suppose that the arithmetically definable sets were themselves arithmetically definable. Then there must be some formula $\varphi(X)$ such that:

$$G \in \mathscr{A} \;\Leftrightarrow\; \mathbb{N}[G] \models \varphi(\dot{G}).$$

- Fix an arithmetic set $G \in \mathscr{A}$. There **must be some finite chunk** p of **information regarding** G such that $\varphi(\dot{G})$ is true in any $\mathbb{N}[H]$ where p is also true of H.

- But there are $2^{\aleph_0}$ many **different** ways of expanding from p into a generic $H \in \mathscr{G}$ and only countably many members of $\mathscr{A}$. So let $H \in \mathscr{G} \setminus \mathscr{A}$ (a generic non-arithmetic set). Then $\mathbb{N}[H] \models \varphi(\dot{G})$ but $H \notin \mathscr{A}$: contradiction.

- In a nutshell, we found a **large** collection $\mathscr{G}$ of sets that could be **controlled** with a finite amount of information.
- The **size** of the collection ensured that we could **adjoin** an element which would lead to a contradiction.

How was it done?

- ***Adjunction*** of a new element to the ground model
- ***Control*** of the augmented model

Adjunction

Adjunction of a new element

- We need to ensure that the element G:
  - **exists**;
  - **exhibits the property** we are seeking; and
  - **is not a member** of the ground model $\mathscr{M}$.

Control

Control of the shape of the new model

- We need to ensure that the generic extension $\mathscr{M}[G]$ **remains a model** of our desired theory.
- We need to ensure that the generic extension doesn't get too complex; that it **remains within reach** of the ground model.

Different types of forcing

- **Simple adjunction** or **closure adjunction**.
  - In **arithmetic** simply adds a new new class to the model,
  - while set theory and second order arithmetic require the domain to augmented so that it satisfies certain **closure conditions**.

A note on our perspective in the proof

- The proofs are all done from a standpoint with sufficient theoretic vantage to be able to talk about both the **ground model**, $\mathscr{M}$, and its **generic extension**, $\mathscr{M}[G]$.
- We shall assume that this viewpoint is that of **set theory**, ZFC.
- Thus, in a sense, there are three points of view involved here:
  - our god-like point of view;
  - the **ground model** $\mathscr{M}$; and
  - that of the **generic extension** $\mathscr{M}[G]$.

A couple of useful *toy* theories

- **Second order arithmetic**: This is the theory of arithmetic with an additional sort of variable for *classes* of natural numbers.
  - The intended model is just the standard model of arithmetic with its powerset, so to speak, added on top.
- **Third order arithmetic**: This is just second order arithmetic with variable for *families* of classes of natural numbers added to the language.
  - Its intended model is obtained by taking the intended model of second order arithmetic and adding the powerset of the classes of naturals onto the top.

Remark about the strategy

- In the examples above, we **adjoined** a new element to a model in such a way that the new model is **still a model of our desired theory**.
- But this makes it sound **too easy**!
- The witness G of the property is **not a finite object**, like say a proof, which we can simply provide.
- Nonetheless we can describe G, **up to a point**, in that we know what we want G to be like.
  - For example, when we show that $V \neq L$ we know that we want G to be a real number that is not already a member of M.

An abortive strategy

- This points the way to a **strategy** for adjoining such a G.
- Rather than **actually providing the** G we could show that:
  - that **every such** G **does the things we are looking for**; and
  - **that there is such a** G satisfying our requirements.

Supervaluation

- This sounds similar to **supervaluation** as used with indeterminate predicates.
- We have **some constraints** about an **indeterminate predicate** and we use those to determine what would be the case regardless of how the extension was settled.
  - We quantify over all the possible precisifications and get **super-truth**.

- Perhaps we could **adopt a similar strategy** for adding a generic set.
- For example, we might consider what happens in **all of the models which contain ω_2** many real numbers.

- Unfortunately, **it doesn't work**!
- This is for a **couple of reasons**, which we'll look at in a moment.
- However, **something** about the strategy **is on the right track**.

Problem 1

- Suppose we are using **arithmetic** (as in the example of Addison's theorem).
- We might then consider what happen if we let $\dot{G}$ denote an **arbitrary** set of natural numbers - perhaps constrained in some way.
- But if we do this, the resultant logic is Π^1_1-**complete.**
- The truths require **quantification over classes** of the full domain of the ground model, with no reduction in sight.
- We have **adjunction** but **no control.**

Adding any old object won't work

- Suppose we are in a **model of third order arithmetic** $\mathscr{M}$ which satisfies full comprehension at both class and family levels.
- Let G be an arbitrary family of ω_2 many real numbers. Let us add G to the model $\mathscr{M}$ and **close it under the comprehension axioms** forming $\mathscr{M}[G]$.
- We seem to have **a model of** $\neg CH$ **which is also a model of third order arithmetic** and we seem to have shown that CH is not a consequence of third order arithmetic.
- But the problem with this argument is that **we have no reason to suppose that there is such a** G**.** Indeed proving the existence of such a set is the whole point of the proof.
- We have **control** without **adjunction**.

Summary

PROBLEM: Can we adjoin an element and retain control?

Focusing on the problem

- So we want to add a more restricted class of sets which allow us to **adjoin** a new element in a **controlled** fashion.
- To do this we are going to introduce the notion of **forcing**.
- Forcing and generic sets are defined **together**. We shall see that:
  - Forcing ensures that we we have sufficient **control** when it comes to **adjoining** the generic element.

But how?

- The **glaring question** is "**how** does the forcing relation do this?"
- The following **quote from Cohen** might help motivate matters:

  ... the set G will not be determined completely, yet properties of G will be completely determined on the basis of very incomplete information about G. ***I would like to pause and ask the reader to contemplate the seeming contradiction in the above.*** *This idea as it presented itself to me, appeared so different from any normal way of thinking, that I felt it could have enormous consequences. On the other hand, it seemed to skirt the possibility of contradiction in a very perilous manner.* [**?**]

- So the idea is to leverage **control** by making incomplete information about G be sufficient to determine facts about $\mathscr{M}[G]$.
- It's a bit like a situation in a game where we are in a position which allows us to **force** our opponent to move in a particular way.
- The trick is to define such a game.

- Let's suppose we are **adding a Cohen real** and that we are using the theory of second order arithmetic.
- We shall represent this **incomplete information** about G using a partial order $\mathbb{P}$, which, in a sense, contains all the different ways we could have gone about approximating G.

- We can **represent** G as a **set of natural numbers**.
- Then, the **partial information** about prospective generic elements would consist of **lists of statements about membership** in G.
- So we might have $\{1 \in G,\ 2 \notin G,\ 56 \in G\}$.
- Or more **tractably**, we might represent this as a **partial function** from ω to 2:

$$\{\langle 1,1\rangle, \langle 2,0\rangle, \langle 56,1\rangle\}$$

- We shall **denote** these **partial functions** by $p, q, \ldots$; and the collection of all these will form the domain P of our partial order $\mathbb{P}$.

- We can then ***order*** these functions in terms **how much information** they provide. If p contains more information about a prospective G than q, then we shall say $p \leq_P q$. Or more formally,

$$p \leq_P q \;\Leftrightarrow\; p \supseteq q.$$

- This direction of the arrow may seem **a little perverse**, but one may think of p as allowing for less possible generic sets to extend it than q.

- This gives us our partial order $\mathbb{P} = \langle P, \leq_P \rangle$, which we will also refer to as a ***condition set*** for obvious reasons.

- We are now in a position to **define what a generic set is** via the concept of forcing.
  - The generic element G provides us with ***adjunction*** while the forcing definition $\Vdash$ gives use the ***control*** we require.
  - The **genius** of the approach is **defining** these things **together**.

Definition

Given a model $\mathscr{M}$, $\langle G, \Vdash \rangle$ is a **generic-forcing pair** if for all $\varphi \in Sent_{\mathscr{L}_G}$,

$$\mathscr{M}[G] \models \varphi \;\Leftrightarrow\; \exists p \in G\ \mathscr{M} \models (p \Vdash \varphi).$$

- This is not the usual way of defining these notions.
- However, since our goal is to **track**, so to speak, the **key concepts** and their **roles** in the proof, I think this approach strikes **closer to the heart** of things.

A specific forcing definition

- We now provide a **specific example** of a forcing relation. We break the definition up into two parts.
- First we provide a definition which deals with the **logical operations** and second we provide one for the **atomic relations**.
  - The forcing relation is **not finished** until we **plug in** an atomic forcing relation.
  - The reason we do this is that we can use the logical forcing relation with a **variety of atomic forcing definitions**.
  - Moreover, examining the logical part on its own will give us a **particular insight** on why it works.

Definition

(*Very general*) Given a partial order $\mathbb{P} \in M$ and an atomic forcing relation $\Vdash\!\vdash$, we define the (*logical*) forcing relation, $\Vdash$, by recursion on complexity of formulae as follows:

- $p \Vdash P\bar{a}$ iff $p \Vdash\!\vdash P\bar{a}$;
- $p \Vdash \neg\varphi$ iff $\forall q \leq p \; q \nVdash \varphi$;
- $p \Vdash \varphi \vee \psi$ iff $p \Vdash \varphi$ or $p \Vdash \psi$; and
- $p \Vdash \exists x\varphi$ iff $\exists x,\ p \Vdash \varphi(x)$.

- The first thing to note is the **clause for negation**. Everything else is much as we'd expect.
- Our goal is to ensure that **anything true in the extension can be ascertained by information from some *finite*** approximation p of the generic element G. The negation clause plays a **crucial role** in ensuring that this will be the case. It says that if there is no approximation extending my current knowledge, in which φ is forced, then we are in a position to force $\neg\varphi$.

Atomic forcing

- We now discuss how to deal with the **atomic cases**. We shall work in second order logic. We shall use lower case letters, $x, y, z, \ldots$ for number variables and upper case variables $X, Y, Z, \ldots$ for class variables.
- One way to do things, would be to add a **new constant symbol** $\dot{G}$ to our language. Then we could say that

$$p \Vdash n \in \dot{G} \iff p(n) = 1.$$

- We certainly want this to be true, but it does not help us much in the case of **closure forcing**. We still don't know what new elements will need to be added to accommodate the adjunction of G.

- So we are going to work, in a sense, the **other way around**. We are going to sort out the **closure conditions up front** and then discover that we have the **means to refer to our generic** G.

- To do this we make use of our condition set $\mathbb{P}$ and define a new type of object which we are going to call a $\mathbb{P}$-**name**.

- We shall denote $\mathbb{P}$-names using upper case Greek letters, $\Gamma, \Delta, \Xi, \Lambda ...$.

- Each of these $\mathbb{P}$-names will represent a particular class in the extended model $\mathscr{M}[G]$. Depending on the condition set $\mathbb{P}$ and ground model $\mathscr{M}$, there may be more classes in $\mathscr{M}[G]$, than in $\mathscr{M}$.

Definition

Γ is a $\mathbb{P}$-*name* if

- for all $\langle n, p\rangle \in \Gamma$, $n \in \omega$ and $p \in P$; and
- if $\langle n, p\rangle \in \Gamma$, then for all $q \leq p$, $\langle n, q\rangle \in \Gamma$.

- Intuitively speaking, our goal is **to tag elements** of ω with **partial information** about G.
- We then want to say that in the extension $\mathscr{M}[G]$, if $\langle n, p\rangle \in \Gamma$ and $p \in G$, then $\mathscr{M}[G] \models n \in \Gamma$.
- So if that piece of partial information p is in the generic G, then it will be true that n is in Γin the generic extension.
- The second condition is designed to **ensure that if we gain more information** about G, say by moving to a stronger $q \leq p$, then the fact that $n \in \Gamma$ is **preserved**.

- In a nutshell, a $\mathbb{P}$-name Γ contains numbers n which are tagged with the information p that would ***force*** them to be members of Γ in the extension.
- This is the **fundamental insight**.

More about G

- So that's how the **forcing definition** works.
- Before we move on, we need to place an **obvious restraint** on what G could be like.
- Intuitively speaking, we want G to contain all the **finite approximations** of our target object, for example a **Cohen real**.
- So we demand that G is **upwardly closed**; i.e., if $p \in G$ and $p \leq q$, then $q \in G$.

- With this in hand, we can then **define the generic extension** $\mathscr{M}[G]$ and the **atomic forcing condition** $\Vdash$.
- $\mathscr{M}[G]$, like $\mathscr{M}$, is intended to be **a model of second order arithmetic**. While we retain the number domain, we shall augment the class domain by providing denotations for each of the $\mathbb{P}$-names.
  - Let $val(\Gamma, G) = \{n \in \omega \mid \exists p \in G \ \langle n, p \rangle \in \Gamma\}$.
  - Let $\mathscr{M}[G]^2 = \{val(\Gamma, G) \mid \Gamma \text{ is a } \mathbb{P}\text{-name}\}$.
- We then define the atomic forcing relation as follows:
  - $p \Vdash n \in \Gamma$ iff $\langle n, p \rangle \in \Gamma$.

Control

- In the next subsection, we shall demonstrate that this **really is a generic-forcing pair**, but for the moment, we can see that we have a device for dealing with provides a **mechanism for dealing with closure forcing**. We shall show that it really does this below.

- However, when we move to $\mathscr{M}[G]$, we still want a **way of referring**
  1. **to G; and**
  2. **and the elements from the ground model $\mathscr{M}$.**

- We need to find a $\mathbb{P}$-names that do this. Fortunately, this is quite easy.

- These are problems of **control**.

2. Finding $\mathscr{M}$ in $\mathscr{M}[G]$

- The numerals clearly refer to the same object in both $\mathscr{M}$ and $\mathscr{M}[G]$ so there aren't any problems here.
- Take a class X from $\mathscr{M}$. Let $\check{X} = \{\langle n, p\rangle \mid n \in X \wedge p \in P\}$. Thus we tag the elements n of X with every element of condition set. Thus **no matter which generic set we form**, X will be denoted by $\check{X}$ in $\mathscr{M}[G]$. We call these ***canonical names***.

1. Talking about G from $\mathscr{M}$'s perspective

- Now let $\dot{G} = \{\langle p, q\rangle \mid q \leq p\}$. The denotation of $\dot{G}$ will then be G.

$$\begin{aligned} val(\dot{G}, G) &= \{n \mid \exists q \in G \ \langle n, q\rangle \in \dot{G}\} \\ &= \{p \mid \exists q \in G \ q \leq p\} \\ &= \{p \mid p \in G\} \end{aligned}$$

- Essentially, we are **tagging each of the conditions** from the partial order with **all of the conditions which are stronger than them**.
- Thus, when we take a particular generic set G, it will be denoted by that name. The **clever thing** about this name (in contrast to *canonical names*) is that its **denotation varies** depending on which generic set we use.

Another constraint on G

- But we still need a **further restriction** on the nature of G. As we have seen above, not just anything will do.
- Clearly, it is a **necessary condition** for G to be generic that for any $\varphi \in \mathscr{L}_G$ there is some $p \in G$ such that $p \Vdash \varphi$ or $p \Vdash \neg\varphi$.
- We add this to our **upward closure** demand.

Proving that a generic G exists

- We shall proceed in a manner similar to a **completeness proof**. Suppose $\mathscr{M}$ is a countable model of second order arithmetic.
- Let $(\varphi_n)_{n\in\omega}$ be an enumeration of the sentences of $\mathscr{L}_G$.
- We define a sequence $(p_n)_{n\in\omega}$ of elements of $\mathbb{P} = \langle P, \leq \rangle$ as follows:
  - Let p_0 be an arbitrary $p \in P$.
  - Let $p_{n+1} = \begin{cases} q & \text{where } q \leq p_n \text{ and } q \Vdash \varphi_n \\ p_n & \text{otherwise.} \end{cases}$
- Note that at any stage $n+1$, if there is no $q \leq p_n$ such that $q \Vdash \varphi_n$, then we clearly have $p_n \Vdash \neg\varphi$, by the negation clause. This how we ensure that the generic set satisfies our requirements.

- Let $G = \{q \in P \mid \exists n \in \omega \; q \geq p_n\}$. G clearly satisfies our condition and it clearly exists.
- It should also be **relatively clear** that $\bigcup G$ is a function with domain ω.
- The **role of the negation clause** should be **particularly apparent** in this construction.

Remarks

- Thus we have ensured the **existence of the generic object**. We started with a model in which all of the finite approximations exist and then add the limit of those constructions.
- Observe that the construction starts from an **arbitrary set** of conditions from $\mathbb{P}$. Thus we can build a generic set from **any** set of conditions.
- So we now have a candidate generic extension $\mathscr{M}[G]$ which contains G and which accommodates closure. We now verify these facts.

We now establish that:

- G and $\Vdash$ form a **generic-forcing pair**;
- $G \notin \mathscr{M}$ (G is new); and
- $\mathscr{M}[G]$ is a model of **second order arithmetic**.

1. Showing this is a generic-forcing pair

We must now show that this is indeed a **generic-forcing pair**.

Theorem

G and $\Vdash$ (as defined above) form a generic-forcing pair.

The following fact is easy to establish and is useful for the proof.

Fact

(Definability) $p \Vdash \varphi$ iff $\mathscr{M} \models (p \Vdash \varphi)$.

Proof.

(of Theorem 9) We must demonstrate that for all $\varphi \in Sent_{\mathscr{L}_G}$,

$$\mathscr{M}[G] \models \varphi \ \Leftrightarrow \ \exists p \in G \ \mathscr{M} \models (p \Vdash \varphi).$$

□

We proceed by induction on the complexity of sentences.

Proof.

(Atomic) Arithmetic sentences are trivial. □

Suppose $\varphi := n \in \Gamma$. Then

$$\begin{aligned} \mathscr{M}[G] \models n \in \Gamma \quad &\Leftrightarrow \quad n \in val(\Gamma, G) \\ &\Leftrightarrow \quad \exists p \in G \ \langle n, p \rangle \in \Gamma \\ &\Leftrightarrow \quad \exists p \in G \ \mathscr{M} \models (\langle n, p \rangle \in \Gamma) \\ &\Leftrightarrow \quad \exists p \in G \ \mathscr{M} \models (p \Vdash n \in \Gamma) \end{aligned}$$

Proof.

(Disjunction) Suppose $\varphi := \psi \vee \chi$. Then $\square$

$$\begin{aligned} \mathscr{M}[G] \models \psi \vee \chi \quad &\Leftrightarrow \quad \mathscr{M}[G] \models \psi \ \vee \ \mathscr{M}[G] \models \chi \\ &\Leftrightarrow \quad \exists p \in G \ \mathscr{M} \models (p \Vdash \psi) \ \ \vee \ \ \exists p \in G \ \mathscr{M} \models (p \Vdash \chi) \\ &\Leftrightarrow \quad \exists p \in G \ \mathscr{M} \models (p \Vdash \psi \vee \chi) \end{aligned}$$

Proof.
(Negation) Suppose $\varphi := \neg\psi$. Then

$$\begin{aligned}
\mathscr{M}[G] \models \neg\psi &\Leftrightarrow \neg\mathscr{M}[G] \models \psi \\
&\Leftrightarrow \neg\exists p \in G \ \mathscr{M} \models (p \Vdash \psi) \\
&\Leftrightarrow \neg\exists p \in G \ p \Vdash \psi \\
&\Leftrightarrow \exists p \in G \ p \Vdash \neg\psi \\
&\Leftrightarrow \exists p \in G \ \mathscr{M} \models (p \Vdash \neg\psi)
\end{aligned}$$

Going ($\rightarrow$) we know that there must be some $p \in G$ which either forces ψ or its negation. Then since there is no $p \in G$ which forces ψ, we see that there is a $p \in G$ which forces $\neg\psi$. Going ($\leftarrow$) we have the consistency of the forcing relation.

$\square$

Proof.

(Quantification) Suppose $\varphi := \exists n \varphi(n)$. Then □

$$
\begin{aligned}
\mathscr{M}[G] \models \exists n \varphi(n) &\Leftrightarrow \exists n\ \mathscr{M}[G] \models \varphi(n) \\
&\Leftrightarrow \exists n\ \mathscr{M} \models (p \Vdash \varphi(n)) \\
&\Leftrightarrow \mathscr{M} \models (p \Vdash \exists n \varphi(n))
\end{aligned}
$$

- So now we know that **any sentence which is true** in $\mathscr{M}[G]$ will be forced by some $p \in G$ and that this fact is verifiable in $\mathscr{M}$. If $\mathscr{M}[G]$ is to be interesting, then $G \notin M$. So although we can verify facts about $\mathscr{M}[G]$ given some $p \in G$, the full information about G is not available inside $\mathscr{M}$.
- The next theorem is a **little stronger** than we require for our forcing definition but it is gives us more semantic grip on forcing.

Theorem

$p \Vdash \neg\neg\varphi$ iff for any $\mathscr{M}$-generic G such that $p \in G$, $\mathscr{M}[G] \models \varphi$.

- Another way of saying that $p \Vdash \neg\neg\varphi$ is to say that for any q which strengthens p there is an r stronger than q which forces φ.
- Intuitively speaking, no matter how much more we learn about G there will always be a way for φ to be forced.
- In this situation, the theorem above tells us that in all the generic extensions $\mathscr{M}[G]$ whose generic element G contains p it will be the case that φ.
- This is much more like **supervaluation**. We are quantifying over different precisifications of our **partial knowledge** about G.

- Historically, this kind of $\neg\neg$ forcing was known as *weak forcing*, whereas our original definition was known as *strong forcing*.
- From now on we shall use weak forcing instead of strong forcing, which we shall denote as $\Vdash_w$.

2. $G \notin \mathscr{M}$

- Now we show that G was **not in the ground model** $\mathscr{M}$.
- But first we prove a little claim.

Fact
For all $p \in P$ there is some $q \leq p$ such that $p \notin G$.

Proof.
To see this, observe that if $p \notin G$, the claim is trivial. So suppose $p \in G$. Then since p is a finite partial function, there will be some n such that $n \notin dom(p)$. Fix such an n. Then for exactly one $m \in \{0,1\}$, $p \cup \{\langle n, m\rangle\} \in G$. Thus $q = p \cup \{\langle n, 1-m\rangle\} \in E$ and clearly $q \leq p$. $\square$

Theorem

$G \notin M$.

Proof.

For *reductio,* suppose $G \in M$. Let $E = P \backslash G$ and let $\check{E}$ be its name. Now consider the sentence $\varphi := \exists r(r \in \dot{G} \wedge r \in \check{E})$. Then by definition, there must be some $p \in G$ such that $p \Vdash \varphi$ or $p \Vdash \neg\varphi$. Clearly, the first condition cannot hold. Thus $p \Vdash \neg\varphi$.

By the claim above, we know that there must be some $q \leq p$ such that $q \in E$. Fix such q and let us extend q to a different generic set H. Thus $\mathscr{M}[H] \models q \in \dot{G} \wedge q \in \check{E}$ and $\mathscr{M}[H] \models \exists r(r \in \dot{G} \wedge r \in \check{E})$; i.e., $\mathscr{M}[H] \models \varphi$. But since $q \leq p$, we also have $p \in H$; and since $p \Vdash \neg\varphi$, we have $\mathscr{M}[H] \models \neg\varphi$: contradiction. $\square$

- To put it briefly, we exploited the power of **partial information over generic sets** to push us into **contradiction**.
- Note the use of the **moving name** $\dot{G}$ and the **fixed name** $\check{E}$. This is the key to the proof.
- This result can be **generalised to any partial order** $\mathbb{P} = \langle P, \leq \rangle$ which has the property that for any $p \in P$ there are $q, r \leq p$ such that q and r have no common extension.
- While the argument above has a very syntactic flavour we should also note an **interesting relationship with Cantor's theorem**.

3. And it's still a model of second order arithmetic

- We must verify that the comprehension axiom is still satisfied in $\mathscr{M}[G]$.
- We shall suppose that we have full second order comprehension; i.e., $\Pi^1_\infty - CA_0$.

Theorem

$\mathscr{M}[G] \models \forall\bar{y}\forall\bar{Y}\exists X\forall n(n \in X \leftrightarrow \varphi(n, \bar{y}, \bar{Y})$ *where* X *does not occur free in* φ.

Proof.

For simplicity, let us ignore the $\bar{y}$ and $\bar{Y}$. Thus given formula $\varphi(n)$, we must find a $\mathbb{P}$-name which denotes its extension in $\mathscr{M}[G]$. $\square$

Proof.

Let $\Delta = \{\langle n, p\rangle \mid p \Vdash \varphi(n)\}$.

We claim Δ suffices. Take an arbitrary $n \in val(\Delta, G)$. Then $\square$

$$\begin{aligned}
\mathscr{M}[G] \models n \in \Delta \quad &\Leftrightarrow \quad \exists p \in G\ \mathscr{M} \models (p \Vdash n \in \Delta) \\
&\Leftrightarrow \quad \exists p \in G\ p \Vdash n \in \Delta \\
&\Leftrightarrow \quad \exists p \in G\ \langle n, p\rangle \in \Delta \\
&\Leftrightarrow \quad \exists p \in G\ p \Vdash \varphi(n) \\
&\Leftrightarrow \quad \exists p \in G\ \mathscr{M} \models (p \Vdash \varphi(n)) \\
&\Leftrightarrow \quad \mathscr{M}[G] \models \varphi(n).
\end{aligned}$$

- So we've now provided a description of **how generic sets are used** and provided a **high level definition of what it means** to be a generic set. These are both, so to speak, **general** parts of the story.
- We now look to some **more specific properties** of generic sets.

Properties of a generic set

- We focus, specifically on sentences which are **about the generic set** G itself. We shall again, limit ourselves to the case where we are adding a Cohen real. We shall use **weak forcing** as it is more convenient.
- Let's consider two types of facts about generic sets - sentences which are:
  - true in **every** generic extension (*generically necessary*); and
  - true in **some** generic extension (*generically possible*).

Fact

$1 \Vdash_w \exists n \; n \in \dot{G}$.

- First, we note that 1 is just the **top element** of the partial order $\mathbb{P}$.
- In the case of building a Cohen real, this is just the **empty set of conditions**, which does nothing to pin down a particular G.

Fact

$1 \Vdash_w \exists n \; n \in \dot{G}$.

- Consider $1 \Vdash_w \exists n \; n \in \dot{G}$. By definition, this means that $1 \Vdash \neg\neg\exists n \; n \in \dot{G}$; and thus, $\forall p \exists q \leq p \exists n \; q \Vdash n \in \dot{G}$. Now we observe that for some q to force $n \in \dot{G}$, all we need is for q to contain that condition. So no matter which state p we are at, it is always possible to find some n which p has nothing to say about and then take a stronger condition q which contains the condition that $n \in \dot{G}$.
- So this is an example of fact about G which is true in **every** generic extension.

Fact

$1 \nVdash_w 17 \in \dot{G}$.

- To see this is false, we just need some p such that every strengthening $q \leq p$ is such that $q \nVdash 17 \in \dot{G}$. Clearly any p which contains the condition $17 \notin \dot{G}$ will suffice for this.
- So this is an example of a fact about G which is **not true in every** extension. It will however, be true in **all those generic extensions which contain the condition** $17 \in \dot{G}$.

- More generally, we can consider the following:

Theorem
$1 \Vdash_w \exists n \in \dot{G}\ \varphi(n)$ *iff* $1 \Vdash_w \forall m \exists n \geq m (n \in \dot{G} \wedge \varphi(n))$.

- This tells us that there is an element n of G such that $\varphi(n)$ in every generic extension iff there are are infinitely many of such elements in every model.

Theorem
$\forall m \exists n \geq m\ \varphi(m)$ iff $1 \Vdash_w \exists n(n \in \dot{G} \wedge \varphi(n)$ for formulae from $\mathscr{L}$ (i.e., formulae without G whose only parameters are from the ground model).

- This tells us that for properties $\varphi(x)$ definable without the aid of parameters from the generic extension $\mathscr{M}[G]$, if infinitely many elements satisfy $\varphi(x)$, then there will be some element of G which satisfies $\varphi(x)$ in every generic extension $\mathscr{M}[G]$.
- The converse is perhaps more interesting. If only finitely many objects satisfy $\varphi(x)$, then there will be a generic extension in which no $n \in \omega$ satisfies $\varphi(x)$.

- This should give us some insight into the nature of generic sets.
  - By ensuring that they only essentially satisfy properties enjoyed by infinitely many objects, we ensure that there they have no distinctive properties.
  - They are **generic.**

Not just any G will do!

- We now observe that if we **do not use a generic set**, then the **truth lemma will fail**.
- For example we might take a G such that $\bigcup G : \omega \to \omega$ where $n \mapsto 0$ for all n. This means that G is **empty.**
- This is clearly **not generic**.
- Moreover since there will be no $p \in G$ and no $n \in \omega$ such that $p_n = 1$ we can see that $\forall p \leq 1 \; p \nVdash \exists n \; n \in \dot{G}$. Thus $1 \Vdash \neg \exists n \; n \in \dot{G}$. But this contradicts our claim above.

- This confirms our observations regarding **supervaluation**.
- Although there is a similarity, **generic sets** are **not merely arbitrary**.

A bit like a completeness proof

- There is another sense in which this proof is **like a completeness proof**.
- It provides us with a **reduction** of a *prima facie* **complex set**.
- In the case of a completeness proof we show that:

$$\forall \mathscr{M} \ \mathscr{M} \models \varphi \quad \Leftrightarrow \ \vdash \varphi.$$

- The left hand side involves **universal quantification over objects of arbitrary cardinality**.
- The right hand side is a **semi-recursive set**, meaning any (positive) membership fact can be verified mechanically in a finite amount of time.

- Similarly in the case of forcing, we end up showing that:

$$\forall G\ \mathscr{M}[G] \models \varphi \quad \Leftrightarrow \quad \exists p \in G\ \mathscr{M} \models (p \Vdash \varphi).$$

- The right hand side of this involves **quantification over objects** which are not (in any interesting cases) even elements of $\mathscr{M}$. We have no reason to think we could say anything about such a relation inside $\mathscr{M}$.

- However, the forcing relation is designed in such a way that facts about such a model **are available in** $\mathscr{M}$, provided *per impossible* we had access to information about G.

- Of course, facts about G are not available in $\mathscr{M}$, but this **lever** is all we need.
- This is another important feature of the **control** of forcing.

Some generic remarks

- The big message of the talk is that forcing is all about **adjunction** and **control**.
  - We **adjoin** a **generic set** G; and
  - the **forcing** relation $\Vdash$ gives us the **control** we require.

References I