Constraints, Dirac observables and chaos

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PoT Workshop @ LMU Munich
4 July 2015

based on:
PH, M. Kubalova and A. Tsobanjan, PRD **86** 065014 (2012); and
B. Dittrich, PH, T. Koslowski and M. Nelson (to appear soon)

GR and 'observables'

General Relativity is a gauge theory

⇒ physical observables should be diffeomorphism invariant

- highly non-local [Torre '93]
- coordinate independent ⇒ dynamics?

in the canonical formulation

- observables should commute with constraints ⇒ Dirac observables as 'constants of motion'
- dynamics relationally ⇒ 'evolving constants of motion' [Wheeler 60's; Rovelli 90's; Dittrich '06,'07......]

either way: important for quantum theory

⇒ notoriously difficult to construct

Relational dynamics à la free particle

Free particle in $\mathbb{R}^2$ with fixed energy

$$C = p_1^2 + p_2^2 - E \simeq 0$$

clearly integrable:

- indep. Dirac observables: p_1 and $L_3 = x_1 p_2 - x_2 p_1$
- relational Dirac observables, choosing x_1 as 'clock'

$$x_2(\tau) = \frac{p_2}{p_1}(\tau - x_1) + x_2 = \mathrm{sgn}(p_2)\frac{\sqrt{E - p_1^2}}{p_1}\,\tau - \frac{L_3}{p_1}$$

- $\{x_2(\tau), p_2\} = 1$ and parametrize reduced phase space

BUT: GR a chaotic system?

Plenty of evidence that a generic general relativistic spacetime chaotic:

- Newtonian $N \geq 3$ body problem chaotic
- $k = 1$ FRW with min. coupled massive scalar chaotic [Page '84; Cornish, Shellard '98; Belinsky, Khalatnikov, Grishchuk, Zeldovich '85]
- Mixmaster (Bianchi IX) universe [Misner '69; Cornish, Levin '97; Motter, Leterlier '01]
- BKL conjecture: generic cosmological solution features chaotic oscillations [Belinsky, Khalatnikov, Lifshitz '70]
- a generic dynamical system is chaotic

Chaos and constants of motion

integrable (unconstrained) systems:

- N (smooth) constants of motion $F_1, \ldots, F_n$ for $2N$-phase space
- if $\{F_i, F_j\} = 0$, the F_i form surface

$$M_F \simeq T^k \times \mathbb{R}^{N-k}$$

non-integrable (unconstrained) systems:

- no global (smooth) constants of motion other than H exist

$\Rightarrow$ trajectories lie on $(2N-1)$-dim. energy surface

- various characterizations:
  - ergodic
  - chaotic
  - ... $\Rightarrow$ distinction unimportant for us, important: non-integrabiliity
- non-integrability generic, $\exists$ concrete theorems for absence of constants of motion [Arnold, Kozlov, Neishtadt book '07]

Dirac observables and chaos? ("Newman's worry" [according to Smolin '01])

3 notions of non-integrability for constrained systems:

1. strong: $\nexists$ 'global' constants of motion in kin. phase space
2. weak: $\nexists$ (global, smooth) Dirac observables other than constraint(s)
3. reduced: non-integrability on reduced phase space (if exists)

non-integrability generic, thus [PH, Kubalova, Tsobanjan '12]:

- smooth Dirac observables (probably) do not exist for full GR

$\Rightarrow$ physical DoFs do not satisfy (Poisson) algebraic structure

- in addition: no good 'clocks' in chaotic systems

$\Rightarrow$ what are the repercussions for quantum gravity?

Are we fooled in our understanding of GR by explicit (integrable) solutions?

Remarkably, this issue has been ignored!

difference:
unconstrained: do not need to solve dynamics
constrained: need to solve dynamics
to access physical DoFs

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Toy model: free particles on a circle

Compactify free dynamics: $x_i + 1 \sim x_i$, $i = 1, 2 \Rightarrow$ conf. manf. $\mathcal{Q} \simeq T^2$

$$C = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - E \approx 0$$

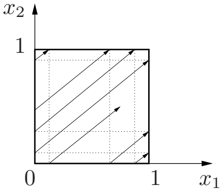
- solutions to EoMs (n_i winding number in x_i)

$$\begin{aligned} x_1(t) &= \frac{p_1}{m_1} t + x_{10} - n_1 \left(\frac{p_1}{m_1} t + x_{10} \right) \\ x_2(t) &= \frac{p_2}{m_2} t + x_{20} - n_2 \left(\frac{p_2}{m_2} t + x_{20} \right) \end{aligned}$$

if:

$\frac{m_2}{m_1} \frac{p_1}{p_2} \in \mathbb{Q}$: resonant torus, periodic orbits

$\frac{m_2}{m_1} \frac{p_1}{p_2} \notin \mathbb{Q}$: non-resonant torus, ergodic orbits



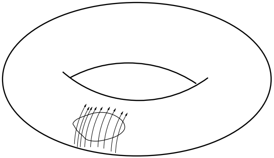
Absence of sufficiently many Dirac observables [Dittrich, PH, Koslowski, Nelson to appear]

- momenta p_i are Dirac observables
- $\exists$ smooth Dirac observables $F(p_i; x_1, x_2)$ with $\partial_i F \neq 0$?

NO: F constant on trajectories must be discontinuous in x_i

- trajectories on non-resonant torus fill it densely

$\Rightarrow$ F takes every value in every neighbourhood (of non-resonant torus)



- hence: ergodicity destroys full integrability

$\Rightarrow$ no reduced phase space, no (sufficient) algebra of observables

- even worse: space of solutions
  1. non-Hausdorff
  2. not a manifold

- failure of Marsden-Weinstein reduction
- model is non-chaotic (topol. entropy zero)

Generalization of Dirac observables

- can still have gauge invariant 'observables', however, either
  1. global and discontinuous
  2. local [Bojowald, PH, Tsobanjan '11a; '11b]

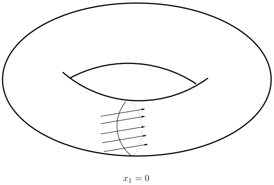
$\Rightarrow$ chaos can be 'observed'

- also relational dynamics still meaningful, albeit implicitly
- e.g.: choose x_1 as 'clock', obtain relational 'observable'

$$x_2(\tau) = \frac{m_1}{m_2}\frac{p_2}{p_1}\left(\tau - x_1 + n_1(\tau, x_2(\tau), x_1, x_2)\right) + x_2 - n_2(\tau, x_2(\tau), x_1, x_2)$$

resonant torus: finitely many solutions

non-resonant torus: 'densely many' solutions



- but: locally, explicit solutions exist on each branch (for fixed n_1, n_2)

Quantization?

1. reduced quantization ×
2. 'standard' Dirac quantization ×
3. Bianca (aka polymer) quantization: discrete topology ✓

Reduced quantization

Reduced quantization

outright impossible since no reduced phase space $\times$

Standard Dirac quantization

- $\mathcal{H}_{\text{kin}} = L^2(S^1 \times S^1)$
- $\hat{p}_i\psi = -i\hbar\partial_i\psi$
- basis:

$$\psi_{k_1,k_2}(x_1,x_2) = \exp(2\pi i k_1 x_1)\exp(2\pi i k_2 x_2), \qquad (k_1,k_2) \in \mathbb{Z}^2$$

- constraint

$$\hat{C} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} - E$$

- solutions to constraint given by k_1, k_2 s.t.

$$k_1^2 + \frac{m_1}{m_2}k_2^2 = \frac{2m_1 E}{\hbar^2}$$

difficult Diophantine problem

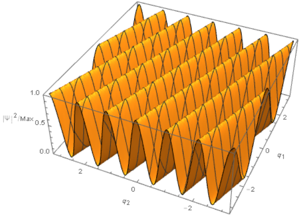
$\Rightarrow$ for $m_1/m_2 \notin \mathbb{Q}$

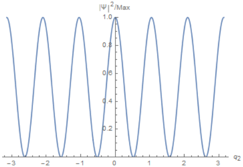
$$0 \leq \dim \mathcal{H}_{\text{phys}} \leq 4$$

- ‘few observables’ $\Rightarrow$ ‘few states’

Sick quantum theory: no semiclassics

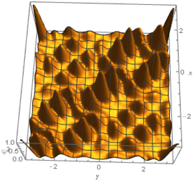
$\dim \mathcal{H}_{\rm phys} = 4$:

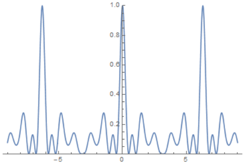




NOT peaked on class. orbit for $m_1/m_2 \notin \mathbb{Q}$ width/separation≈ 1

$\dim \mathcal{H}_{\rm phys} = 12$: $(m_1/m_2 = 1)$





states decohere 'initial' localization at $x_1 = 0$

Quantizing the Bianca (polymer) way: discrete topology

additional 'observables' discontinuous $\Rightarrow$ try discrete topology on T^2

- $\mathcal{H}_{\rm kin}$ given by (uncountable) basis

$$\psi_{x_1',x_2'}(x_1,x_2) = \delta_{x_1',x_1}\delta_{x_2',x_2}$$

and

$$\langle\psi_{x_1',x_2'}|\psi_{x_1'',x_2''}\rangle := \int d\mu_d(x_1,x_2)\,\delta_{x_1',x_1}\delta_{x_2',x_2}\delta_{x_1'',x_1}\delta_{x_2'',x_2} = \delta_{x_1',x_1''}\delta_{x_2',x_2''}$$

states

$$\psi(x_1,x_2) = \sum_{|i|<\infty} c_i\,\delta_{x_1^i,x_1}\delta_{x_2^i,x_2}$$

- no momenta, but translations

$$(R_1^\mu\,\psi)(x_1,x_2) = \psi(x_1+\mu,x_2), \qquad (R_2^\mu\,\psi)(x_1,x_2) = \psi(x_1,x_2+\mu)$$

$\Rightarrow$ $p_i^2/2$ replaced by

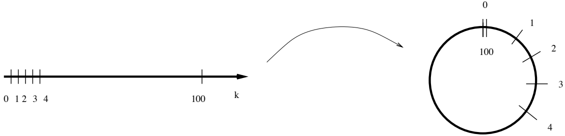
$$S_i^\mu := -\frac{\hbar^2}{2\mu^2}(R_i^{+\mu} + R_i^{-\mu} - 2)$$

- constraint

$$\hat{C}^\mu = S_1^\mu + S_2^\mu - E$$

Large physical Hilbert space and enough observables

Bohr compactification:



- eigenstates and eigenvalues R_i^μ

$\mu \in \mathbb{Q}$: discrete

$$\phi_{x',k}(x) = \frac{1}{\sqrt{q}} \sum_{l=0}^{q-1} e^{2\pi i k l \mu} \, \delta_{x'+l\mu,x}, \qquad \{e^{2\pi i k \mu} | k = 0, \ldots, q-1\}$$

$\mu \notin \mathbb{Q}$: continuous $\rho \in [0,1)$

$$\phi_{x',\rho}(x) = \sum_{l \in \mathbb{Z}} e^{2\pi i l \rho} \, \delta_{x'+l\mu,x}, \qquad \{e^{2\pi i \rho} \in \mathrm{U}(1)\}$$

- spectrum of constraint $\hat{C}^\mu$

$\mu \in \mathbb{Q}$:

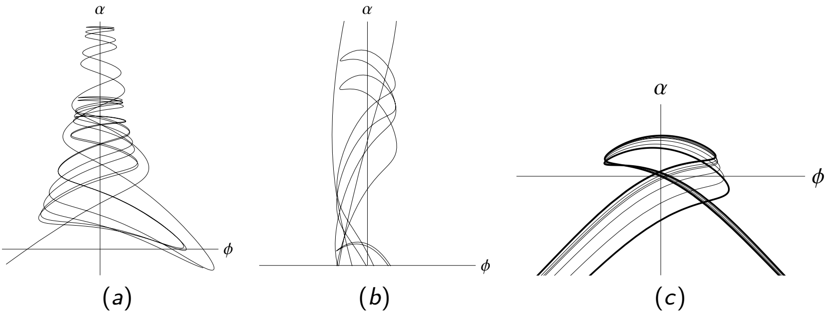
$$\{\frac{\hbar^2}{\mu^2} (2 - \cos(2\pi k_1 \mu) - \cos(2\pi k_2 \mu)) - E | k_1, k_2 = 0, \ldots, q-1\}$$

$\mu \notin \mathbb{Q}$:

$$\{\frac{\hbar^2}{\mu^2} (2 - \cos(2\pi \rho_1) - \cos(2\pi \rho_2)) - E | \rho_1, \rho_2 \in [0,1)\}$$

$\Rightarrow$ get inf. dim. $\mathcal{H}_{\mathrm{phys}}$ and sufficiently many Dirac observables!!!

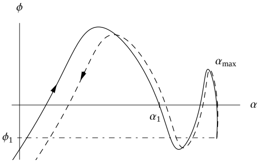
closed FRW with massive scalar



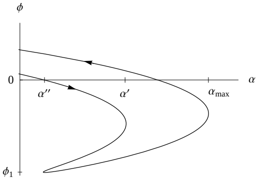
(a) typical solution, (b) close-up on (a), (c) defocussing of nearby trajectories in turning region

- model chaotic and non-integrable **[Page '84, Cornish, Shellard '98; Belinsky, Khalatnikov, Grishchuk, Zeldovich '85]**
- solution space has fractal structure **[Page '84, Cornish, Shellard '98]**
- strong defocussing of classical solutions near α_{max}
- devoid of good clocks **[PH, Kubalova, Tsobanjan, '12]** $\Rightarrow$ problem for 'standard' QT

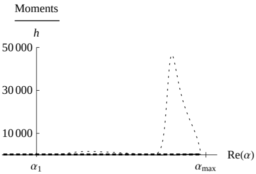
Breakdown of quantum relational dynamics



classical solution

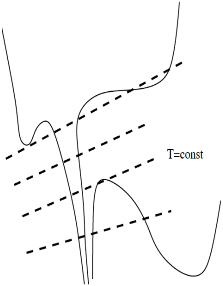


close-up on α_{max}



moments in initial α–time

- generic classical trajectory has structure below chosen quantum scale
- semiclassicality generically breaks down in region of maximal expansion ('too much structure' + defocussing) **[PH, Kubalova, Tsobanjan '12; Kiefer '88]**
- any clock 'bad' in this region, no clock change possible $\Rightarrow$ relational evolution breaks down **[PH, Kubalova, Tsobanjan '12]**



Conclusions

- Chaos destroys integrability and existence of smooth Dirac observables

$\Rightarrow$ probably no smooth Dirac observables and red. phase space for full GR

- but: generalized discontinuous ‘observables’
- serious problem for standard constraint quantization

what do we do?

- smear out energy to $[E-\epsilon, E+\epsilon]$
- Shape Dynamics, HL gravity do not face this problem since no Hamiltonian constraint
- quantize integrable subsector of GR? $\Rightarrow$ that's cheating!
- wait for Bianca to discretely save the world (aka quantum gravity)

- abolish idea of ‘wave function of the universe’ [PH '14]