Changing Observables in Canonical General Relativity from Hamiltonian-Lagrangian Equivalence

J. Brian Pitts
Faculty of Philosophy, University of Cambridge

Problem of Time Workshop
Ludwig-Maximilians-Universität München,
Munich, Germany, 3-4 July 2015

Outline

First-Class Constraints and Gauge?

- Take Poisson brackets among all constraints.
- For Maxwell, Yang-Mills, GR, all PB are 0 identically or using the constraints: "first-class."
- What *exactly* do first-class constraints (FCC) have to do with gauge freedom?
- Original, recently recovering view: *tuned sum* of FC constraints generates gauge transformations, the same ones as in $\mathcal{L}$ (Rosenfeld, 1930; Anderson and Bergmann, 1951; Mukunda, 1980; Castellani, 1982; Shepley et al., 2000).
- *Vs.* Widespread Belief: each FC constraint by itself generates a gauge transformation (Dirac, 1964; Govaerts, 1991; Henneaux and Teitelboim, 1992; Rothe and Rothe, 2010).

No Change *vs.* Common Sense and $G^{\mu\nu} = 8\pi T^{\mu\nu}$

- (1) The world—it's changing, falsifying Hamiltonian GR?
- (2) Einstein's equations—most solutions have change (time dependence in all coordinate systems).
- Those that don't change are "stationary": they have a "time-like Killing vector field," which tells how to define time such that the space-time metric is independent of it.
- Time-like Killing vector field ξ^{μ} is gauge-invariant form of $\partial g_{\mu\nu}/\partial t = 0$.

Testing Widespread Belief for Maxwell

- For Maxwell, Yang-Mills, and GR, Dirac's argument fails (Pons, 2005; Pitts, 2014b; Pitts, 2014a).
- Pons: Dirac stops too soon. At 2nd infinitesimal order, secondary constraints appear to help the primaries generate a gauge transformation, hence gauge generator G (Pons, 2005).
- Trouble at 1st order if one doesn't cancel by subtracting two trajectories (Pitts, 2014b).
- Primary: $\delta A_\mu(t,x) = \{A_\mu(t,x), \int d^3y p^0(t,y)\xi(t,y)\} = \delta^0_\mu \xi(t,x)$.
- $F_{\mu\nu}[A+\delta A] - F_{\mu\nu}[A] = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = \partial_\mu \xi \delta^0_\nu - \partial_\nu \xi \delta^0_\mu \neq 0$.
- $\vec{B}$ is invariant, but $\vec{E}$ isn't! Gauss's law is spoiled.
- A first-class primary constraint generates a ~~gauge transformation~~ violation of Gauss's law.

What Does a Secondary First-Class Constraint Generate?

- Secondary:
  $\delta A_\mu(t,x) = \{A_\mu(t,x), \int d^3y p^i,_i (t,y)\epsilon(t,y)\} = -\delta^i_\mu \frac{\partial}{\partial x^i}\epsilon(t,x).$
- $\delta F_{\mu\nu} = \partial_\mu(-\delta^i_\nu \frac{\partial}{\partial x^i}\epsilon) - \partial_\nu(-\delta^i_\mu \frac{\partial}{\partial x^i}\epsilon) = \delta^i_\mu \partial_\nu \partial_i \epsilon - \delta^i_\nu \partial_\mu \partial_i \epsilon.$
- $\delta F_{0n} = -\delta \vec{E} = \delta^i_0 \partial_n \partial_i \epsilon - \delta^i_n \partial_0 \partial_i \epsilon = -\partial_n \partial_0 \epsilon \neq 0.$
- Gauss's law is spoiled again.
- Not a gauge transformation (Pitts, 2014b).
- *Vs.* Dirac's conjecture.
- Making a right out of two wrongs: $\delta \vec{E}(\xi)$, $\delta \vec{E}(\epsilon)$ can be cancelled out by setting $\xi = -\dot{\epsilon}$.
- *Team* is gauge generator $G = \int d^3y[-p^0(y)\dot{\epsilon} + p^i,_i \epsilon(y)]$ (Anderson and Bergmann, 1951; Castellani, 1982).

Hamiltonian-Lagrangian Equivalence Sought and Found

- So GR H is a sum of things that change the world, not redescribe it.
- Thus no reason to doubt change.
- Long series of works including ((Pons and Shepley, 1995; Pons et al., 1997; Pons and Shepley, 1998; Shepley et al., 2000; Pons and Salisbury, 2005; Pons et al., 2010)).
- "We have been guided by the principle that the Lagrangian and Hamiltonian formalisms should be equivalent. . . in coming to the conclusion that they in fact are." (Pons and Shepley, 1998, p. 17)
- If observables inconvenient (Yang-Mills) or controversial (GR), test for preserving Hamilton's equations by (quasi-) invariance of canonical action $S_H = \int d^4x(p\dot{q} - H)$.

First-Class Constraints and the Hamiltonian Action

- Changing S by only BT if, only if a gauge transformation.
- Canonical $\mathcal{L}_H$ equivalent to $\mathcal{L}$; p^i are auxiliary fields. No p^0.
- $\mathcal{L}_H = p^i \dot{A}_i - \mathcal{H}_c = p^i \dot{A}_i - \frac{1}{2} p^{i2} - p^i A_{0,i} - \frac{1}{4} F_{ij}^2$. (Maxwell)
- PB of smeared primary FC constraint with $\mathcal{L}_H$: $\{\int d^3y p^0 \xi(t,y), [p^i(x)\dot{A}_i - \mathcal{H}_c]\} = -\xi p^i{}_{,i} \not\equiv div$: not gauge!
- PB of secondary: $\{\int dt \int d^3y \epsilon(t,y) p^i{}_{,i}\, , \int d^3x [p^j(x)\dot{A}_j - \mathcal{H}_c]\}$ $= \int dt d^3x - p^i{}_{,i}(x)\dot{\epsilon}(t,x) \not\equiv BT$: not gauge!
- Keeping $\vec{E}$ invariant, S_H (quasi-)invariant, agree and give G.
- Adding primary & secondary with *related* coefficients gives team, the gauge generator $G = \int d^3x(-p^0\dot{\epsilon} + \epsilon(x) p^i{}_{,i})$.
- G changes S_H by BT only, a gauge transformation.

(Homogeneous) GR Action vs. First-Class Constraints

- $L_H = p\dot{q} - H = p\dot{N} + \pi^{ij}\dot{h}_{ij} - H_p = \pi^{ij}\dot{h}_{ij} - N\mathcal{H}_0.$
- What does primary FCC do to L_H *via* Poisson bracket? $\{\xi p, \pi^{ij}\dot{h}_{ij} - N\mathcal{H}_0\} = \{\xi p, -N\mathcal{H}_0\} = \xi\mathcal{H}_0 \not\equiv div$: not gauge!
- Secondary FCC, the "Hamiltonian constraint" $\mathcal{H}_0$, supposedly generates time evolution *and* gauge transformations.
- In fact $\mathcal{H}_0$ generates *neither*: $H_p = N\mathcal{H}_0 + \dot{N}p$ does former, G does latter, but $\mathcal{H}_0$ is the *star player on both teams*.
- To avoid $\dot{N}$, use $\epsilon^{\perp}$ $(= N\xi^0 = -\xi^{\mu}n_{\mu})$ as primitive (0 PB).
- $\{\epsilon(t)\mathcal{H}_0, \pi^{ij}\dot{h}_{ij} - N\mathcal{H}_0\} = \{\epsilon\mathcal{H}_0, \pi^{ij}\dot{h}_{ij}\} - 0 =$ $\{\epsilon\mathcal{H}_0, \pi^{ij}\}\dot{h}_{ij} + \{\epsilon\mathcal{H}_0, \dot{h}_{ij}\}\pi^{ij} = \epsilon\dot{\mathcal{H}}_0 - \frac{\partial}{\partial t}\left(\epsilon\pi^{ij}\frac{\partial\mathcal{H}_0}{\partial\pi^{ij}}\right) \not\equiv div!$

Star Players in Two Team Sports: Bo Jackson, $\mathcal{H}_0$ and $\mathcal{H}_i$

- "He is the only athlete to be named an All-Star in two major American sports." (Wikipedia, 2015)
- But was not *alone* a football or baseball team.
- $\mathcal{H}_0$ and $\mathcal{H}_i$ are analogous.

Figure: Bo Jackson (Kingdom Magazine, nd); $\mathcal{H}_0$ and $\mathcal{H}_i$ not pictured.

Change of Time Coordinate in Repaired Hamiltonian GR

- Unclarity where time coordinate transformations went in Hamiltonian GR (Kuchař, 1986; Belot and Earman, 2001), due to Dirac (Dirac, 1958).
- But it is there (on-shell) if done rightly using G (Fradkin and Vilkovisky, 1977; Castellani, 1982; Shepley et al., 2000; Pitts, 2014a).
- Tune coefficients of primary and secondary FCC to get G.
- G changes $p\dot{q} - H$ by $\frac{\partial}{\partial t}\left(\epsilon\left[\mathcal{H}_0 - \pi^{ij}\frac{\partial\mathcal{H}_0}{\partial\pi^{ij}}\right]\right) \equiv div$: gauge!
- Make Hamiltonian change in GR match ordinary change.
- No essential difficulty expected with full GR with matter.

Testing Widespread Belief in Vacuum GR: FC Primaries

- Lagrangian constraints are time-space and time-time Einstein's equations, Gauss-Codazzi relations embedding space into space-time: $D^i K_{ij} - D_j K = 0$, $K^{ij} K_{ij} - K^i_i K^j_j - R$.
- $K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - D_i \beta_j - D_j \beta_i)$.
- To get H, find primaries $p_0 =_{df} \frac{\partial \mathcal{L}}{\partial N,_0} = 0$ and $p_i =_{df} \frac{\partial \mathcal{L}}{\partial \beta^i,_0} = 0$.
- If FC primaries generate gauge, they will preserve embedding of space into space-time.
- p_0 varies only N; p_i varies only shift β^i.
- They spoil Lagrangian constraints.
- Phase space constraints $\mathcal{H}_0 = 0$ and $\mathcal{H}_i = 0$ still hold, because mere auxiliary field π^{ij} hasn't changed but K_{ij} has!

- Vary N: $\{\int d^3y \epsilon(y) p(y), D_i(K^i_j - \delta^i_j K)(x)\} = D_i[(K^i_j - \delta^i_j K)\epsilon N^{-1}](x) \neq 0.$
- $\{\int d^3y \epsilon(y) p(y), K^{ij}K_{ij} - K^i_i K^j_j - R(x)\} = 2\epsilon(x) N^{-1}(K^{ij}K_{ij} - K^2) \neq 0.$
- p spoils all 4 of the constraints in Einstein's equations.
- Vary β^i. $\{\int d^3y \epsilon^i(y) p_i(y), D_l(K^l_j - \delta^l_j)(x)\} = D_i(\frac{1}{2N} D_j \epsilon^i) + D^i(\frac{1}{2N} D_i \epsilon_j) - D_j(N^{-1} D_i \epsilon^i) \neq 0.$
- $\{\int d^3y \epsilon^l(y) p_l(y), K^{ij}K_{ij} - K^2 - R(x)\} = (D_j \epsilon^i) \frac{2}{N}(K^j_i - \delta^j_i K)(x) \neq 0.$
- p_l spoils Lagrangian constraints also.
- Analogs of electromagnetism result (Pitts, 2014b).
- A primary first-class constraint ~~generates a gauge transformation~~ ruins the embedding of space into space-time.

Historical Interlude

- Long was easy to neglect $4-d$ coordinate transformations because 1958 $3+1$ trivialization of primaries (Anderson, 1958; Dirac, 1958) rendered obsolete the original G (Anderson and Bergmann, 1951).
- Dirac discouraged $4-d$ symmetry: drops primaries and even N, β^i! (Dirac, 1958).
- Gauge-fixing $N=1$, β^i (coordinate conditions) also played a role (Pons et al., 2009).
- $3+1$ G finally appeared in 1982 (Castellani, 1982).
- Ongoing process of rethinking what took root during 1958-82.

What Do FC Secondaries $\mathcal{H}_i$ and $\mathcal{H}_0$ Generate?

- $\{h_{ij}(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} = \pounds_\xi h_{ij}(x)$, $\{\pi^{ij}, \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} = \pounds_\xi \pi^{ij}$: coordinate transformation?
- But only of $3 - metric$ and only on one slice.
- $\{\beta^i(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} = 0$, $\{N(x), \int d^3y \epsilon^i(y)\mathcal{H}_i(y)\} = 0$.
- $\{h_{ij}(x), \int d^3y \epsilon^\perp(y)\mathcal{H}_0(y)\} = \epsilon^\perp(2\pi_{ij} - \pi h_{ij})/\sqrt{h} =$ (on-shell) $\delta^\mu_i \delta^\nu_j \pounds_{(\epsilon^\perp n^\alpha)} g_{\mu\nu}(x)$.
- But $\{N(x), \int d^3y \epsilon^\perp(y)\mathcal{H}_0(y)\} = 0$ and $\{\beta^i(x), \int d^3y \epsilon^\perp(y)\mathcal{H}_0(y)\} = 0$: no coordinate transformation, no gauge transformation.
- δK_{ij} from δg_{ij} due to $\mathcal{H}_i$: $\delta K_{ij} = \frac{1}{2N}\frac{\partial}{\partial t}\pounds_\epsilon g_{ij} - \frac{1}{2N}[(\pounds_\epsilon g_{lj})D_i\beta^l + (\pounds_\epsilon g_{il})D_j\beta^l + \beta^m D_m \pounds_\epsilon g_{ij}]$.
- Equivalent to $\delta K_{ij} = \frac{1}{2N}\frac{\partial}{\partial t}\pounds_\epsilon g_{ij} - \frac{1}{2N}[g_{lj}\beta^m \pounds_\epsilon \Gamma^l_{im} + g_{il}\beta^m \pounds_\epsilon \Gamma^l_{jm} + (D_i\beta^l)\pounds_\epsilon g_{lj} + (D_j\beta^l)\pounds_\epsilon g_{li}]$.

FC Secondaries Generate Bad Physical Changes in GR

- $\mathcal{H}_0$ and $\mathcal{H}_i$ do not generate gauge transformations, but spoil Lagrangian constraints, $\frac{4}{10}$ of Einstein's equations, the Gauss-Codazzi relations embedding space into space-time.
- $\delta K_{ij} = \pounds_\epsilon K_{ij}$ (good) $+\not\delta K_{ij}$ (bad).
- $\delta D_i(K^i_j - \delta^i_j K) = \pounds_\epsilon D_i(K^i_j - \delta^i_j K) + D_i(h^{il} \not\delta K_{lj} - \delta^i_j h^{ab} \not\delta K_{ab})$.
- 0th order: $g_{\mu\nu} = diag(-1, 1, 1, 1)$, $N = 1$, $\beta^i = 0$, $g_{ij} = \delta_{ij}$, so $\delta D_i(K^i_j - \delta^i_j K) = \pounds_\epsilon 0 + \partial_i(\delta^{il} \not\delta K_{lj} - \delta^i_j \delta^{ab} \not\delta K_{ab})$.
- $\not\delta K_{ij} = \frac{1}{2}(\dot{\epsilon}_{i,j} + \dot{\epsilon}_{j,i})$.
- Variation in $q - \dot{q}$ momentum constraint is $\frac{1}{2}(\partial_i \partial_i \dot{\epsilon}_j - \partial_j \partial_i \dot{\epsilon}_i \neq 0)$, bad physical change, spoiling 40% of Einstein's equations.

- Remaining PB between secondaries and Lagrangian constraints: $\{\int d^3y \epsilon(y)\mathcal{H}_0(y), D_l(K^l_j - \delta^l_j K)(x)\}$ (completed), $\{\int d^3y \epsilon^k(y)\mathcal{H}_k(y), K_{ij}K^{ij} - K^2 - R(x)\}$ (completed, cross-checked, given below), and $\{\int d^3y \epsilon \mathcal{H}_0(y), K_{ij}K^{ij} - K^2 - R(x)\}$ (completed).
- One can now find $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\}$, $\{G[\epsilon^\perp, \dot{\epsilon}^\perp], D_l(K^l_j - \delta^l_j K)(x)\}$, $\{G[\epsilon^\perp, \dot{\epsilon}^\perp], K_{ij}K^{ij} - K^2 - R(x)\}$,
- $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\}$: completed, gives spatial $\pounds_\xi$.
- $\{\int d^3y \epsilon^k \mathcal{H}_k(y), K_{ij}K^{ij} - K^2 - R\} = -\pounds_\epsilon (K^{ab}K_{ab} - K^2 - R)$ $-\frac{K^{ij} - h^{ij}K}{N} \pounds_{\dot{\epsilon}} h_{ij} - 2\frac{K^{ij}K_{ij} - K^2}{N} \pounds_\epsilon N - 2\frac{K^{ij} - h^{ij}K}{N} \pounds_\epsilon D_i \beta_j +$ $\frac{2}{N}(K^{ij} - h^{ij}K)(D_i\beta^l)\pounds_\epsilon h_{jl} + \frac{2}{N}(K^i_l - K\delta^i_l)\beta^m \pounds_\epsilon \Gamma^l_{im}$.

Gauge Generator G from Tuned Sum in GR

- Poisson bracket of tuned smeared sum of $\mathcal{H}_i$, p_i and p.
- $G[\epsilon^k, \dot{\epsilon}^k] = \int d^3y[\epsilon^k(y)\mathcal{H}_k + (\pounds_\epsilon \beta^k + \dot{\epsilon}^k)p_k + N,_k \epsilon^k p]$ (Castellani, 1982; Pons et al., 2000).
- $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\}$ is somewhat long.
  Highlights:
  variation of $\dot{h}_{ab}$ in K_{ab},
  $\pounds_\xi \Gamma^\alpha_{\mu\nu}$,
  cancellation of $\dot{\epsilon}^i$ terms from different constraints,
  cancellation of all the many terms $(D\vec{\beta})(D\vec{\epsilon})$ terms,
  cancellation of symmetric parts of $\beta D^2 \epsilon$ and of $\epsilon D^2 \beta$,
  cancellation of resulting spatial Riemann terms.
- One gets a Lie derivative as expected:
  $\{G[\epsilon^k, \dot{\epsilon}^k], K_{ij}K^{ij} - K^2 - R(x)\} = -\pounds_\epsilon(K_{ij}K^{ij} - K^2 - R(x))$.

But Is There Change in “Observables”?

- “There are indications that the Hamiltonian of the general theory of relativity may vanish and that all the observables are constants of the motion.” (Bergmann and Goldberg, 1955)
- “No genuine physical magnitude countenanced in GTR changes over time.” (Earman, 2002)
- A problem of space also: spatially constant from $\{O, \mathcal{H}_i\} = 0$ (Torre, 1993).
- Most common definition (though Bergmann said other things too): $\{O, FC\} = 0$ for all FC constraints (Bergmann, 1956; Bergmann, 1961; Earman, 2002).
- Can infer from Widespread Belief.

- Observables constant over time because 0 PB with (alleged) generator of time evolution $\mathcal{H}_0$: problem of time (Earman, 2002).
- That's crazy (Maudlin, 2002). Yes, but what went wrong?
- "Observables" seems to be a technical term, nonlocal and marginally related to observation (Kiefer, 2012; Tambornino, 2012)!
- But inventor Bergmann intended otherwise. "General relativity was conceived as a local theory, with locally well defined physical characteristics. . . . We shall call such quantities *observables*. . . . We shall call *observables* physical quantities that are free from the ephemeral aspects of choice of coordinate system and contain information relating exclusively to the physical situation itself. Any observation that we can make by means of physical instruments results in the determination of observables;" (Bergmann, 1962, p. 250).

Hamiltonian Observables Repaired

- Break down usual definition of observables as supposedly gauge-**in**variant by virtue of having $\mathbf{0}$ Poisson bracket with **each** first-class constraint (Bergmann, 1956; Bergmann, 1961; Earman, 2002).
- Key features of usual definition: **in**variant ($\mathbf{0}$), **each**.
- First Problem: usual definition depends on Widespread Belief.
- $\vec{E}$ is not "observable" in that sense (Pitts, 2014b).
- Replace "**each** first-class constraint" with gauge generator G. Amended: observables are gauge-**in**variant, having $\mathbf{0}$ Poisson bracket with ~~each first-class constraint~~ the gauge generator $G[\xi^\alpha]$ $(\forall \xi^\alpha)$ (Pons et al., 2010).
- Now *electric* and magnetic fields are observable.

Internal *vs.* External Gauge Symmetries

- **In**variance (0 PB) because electromagnetic gauge choice is operationally ineffable. Voltmeter in Coulomb gauge *vs.*. . . ?
- Electromagnetic observables invariant, like propositions in Plato's heaven, unlike sentences in a language (translatable).
- But I cross the Prime Meridian on the way to work.
- In GR G acting on a quantity ϕ gives Lie derivative $\pounds_\xi \phi$, directional *derivative* of ϕ. (Castellani, 1982).
- $\pounds_\xi g_{\mu\nu} = \xi^\alpha g_{\mu\nu},_\alpha + g_{\mu\alpha}\xi^\alpha,_\nu + g_{\alpha\nu}\xi^\alpha,_\mu .$
- Transport term $\xi^\alpha g_{\mu\nu},_\alpha$ differentiates $g_{\mu\nu}$: "external."
- Maxwell, Yang-Mills have no ∂A term in transformation rule for A, hence "internal" gauge symmetry.
- But Bergmann and Dirac made no such distinction for observables (Bergmann, 1956; Bergmann, 1961; Dirac, 1964).

External Covariance *vs.* Internal Invariance for Observables

- Crossing the Prime Meridian on the way to work.

- But I never find myself crossing $A_0 = 0$ surface.

A Classical Tour of the Lie Derivative

- Transport term $\xi^\alpha g_{\mu\nu,\alpha}$ arises from comparing *different* space-time points with *same* coordinate values in *different* coordinate systems (Bergmann, 1949; Schouten, 1954).
- 1 a.m. British Summer Time *vs.* 1 a.m. GMT *an hour later*.
- Mathematically convenient but physically weird fixed coordinate variation $\bar{\delta}A = A'(x; p') - A(x; p)$.
- *C.f.* physically reasonable but mathematically inconvenient fixed point variation $\delta A = A'(x'; p) - A(x; p)$.
- "the commutativity of the operations of [fixed coordinate] $\bar{\delta}$-variation and partial differentiation. . . is the basic reason for our preoccupation with $\bar{\delta}$-variation processes." (Bergmann, 1957, p. 16)
- Difference is transport term.

Figure: Fixed Coordinate Variation: British Summer Time (up) *vs.* GMT

Observables Don't Need 0 PB for External Symmetry

- Mathematical convenience takes precedence over physical meaning in deriving the Lie derivative.
- For *physically* individuated point p (Einstein's point-coincidence argument, relationalism), scalar at p is gauge-invariant (Hoefer, 1996; Maudlin, 2002).
- But G changes coordinates *and* compares different places.
- For scalar field, only second task. Ricci scalar $R(p)$: $\{G, R\}$ =(on-shell) $-\pounds_{\xi} R = -\xi^{\alpha} R_{,\alpha}$.
- If observables need 0 PB with $G[\xi^{\alpha}]$ $(\forall \xi^{\alpha})$, gauge-invariant $R(p)$ is observable only if spatio-temporally constant.
- "*Every quantity in a cosmological theory that is formally an observable should in fact be measurable by some observer inside the universe*" (Smolin, 2001).

Observables Should Vary Locally

- "General relativity was conceived as a local theory, with locally well defined physical characteristics. We shall call such quantities *observables*. (Bergmann, 1962, p. 250).
- Constancy of observables is not an insight, but a *reductio* of a bad definition.
- Wrong requirement of 0 Poisson bracket with G as applied to external symmetries.
- O with $\{O, time\ gauge\ generator\} \stackrel{!}{=} 0$ like unicorns (Kuchař, 1993).
- But Kuchař doesn't apply his argument to space, for which it is equally persuasive.

Observables as 4-d Tensor Calculus All Over Again

- Need Hamiltonian analog of O's being a 4-d *geometric object*: components in coordinate systems with transformation law, hence $\{G, O\} = -\pounds_\xi O$ (on-shell).
- For vectors, tensors, *etc.* ψ, $\xi^\mu \psi,_\mu$ is no vector, tensor, *etc.*
- Lie derivative involves correction terms $g_{\mu\alpha}\xi^\alpha,_\nu + g_{\alpha\nu}\xi^\alpha,_\mu$ from tensor transformation law, fixed point δ-variation.
- Observables as classical vectors, tensors, *etc.*: covariant (having a transformation rule, translatable), not invariant.
- Related claim from Brunetti, Fredenhagen and Rejzner: "The way out is to replace the requirement of invariance by covariance." (Brunetti et al., 2015, p. 3)
- *In*variance applies, if at all, only to non-numerical tensor-in-itself $\mathbf{g} = g_{\mu\nu}\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$, up to transport term.

All Coordinates Are Intrinsic If Any Are

- Scalars from Weyl tensor $C^{\alpha}_{\beta\mu\nu}$ as intrinsic coordinates (Komar, 1955; Gèhèniau and Debever, 1956; Bergmann and Komar, 1960; Pons and Salisbury, 2005).
- "[T]he observables obtained may alternatively be viewed as the metric tensor in a special 'gauge' (i.e., with a special coordinate condition)." (Komar, 1958)
- "Let A^i be the four functionally independent curvature scalars (i.e., four specific and distinguishable scalar functions constructed from the metric tensor and its derivatives). To emphasize that these four functions uniquely and intrinsically identify world points, let us go to the new coordinate system determined by the A^I:

$$\bar{x}' = A^i(x) \tag{2.1}$$

"If we inquire into what the metric tensor looks like in this new coordinate system we find the usual expression:

$$\bar{g}^{ij} = \frac{\partial A^i}{\partial x^m}\frac{\partial A^j}{\partial x^n} g^{mn}. \qquad (2.2)$$

However, we now note that since A^i is a scalar, the $\partial A^i/\partial x^m$ is a covariant vector and therefore $\bar{g}^{ij}$ is component by component a well defined scalar constructed from the metric tensor and its derivatives." (Komar, 1958)

- Weyl scalars are $\sim C^2(g, \partial g, \partial^2 g)$, $\sim C^2$, $\sim C^3$, $\sim C^3$, so admit arbitrary functions of them as intrinsic.
- *Any* coordinate system is intrinsic if one is.
- Transformation rule is needed from one set to another.
- All coordinate systems are preferred, so none is.
- Hence $g^{\mu\nu}$ is observable: $4-d$ tensor calculus.

- Arriving more easily at an attractive result:
- "Indeed, once we have proven that observables can be built for any observer, we can gladly dispose of this [active] construction [involving exponentiating Poisson brackets, with relations to Dittrich and Rovelli] and just take the passive view of diffeomorphism invariance. We simply instruct each observer, having constructed his or her phase space solutions, to transform them to the intrinsic coordinate system! . . . Thus here is the guiding principle: let everyone adopt the same instrinsic coordinates. Once this instruction is implemented all geometric onjects becomes observable!" (Pons et al., 2010)
- By never introducing unphysical primitive point individuation, one doesn't have to overcome it.

Observables and Point Individuation: $R(p)$?

"…general relativity does not…identify the history of a physical universe with a manifold on which are defined a metric and perhaps other fields. The correct statement is that the history of a universe is defined by an *equivalence class of manifolds and metrics under arbitrary diffeomorphisms*[3].

This is a key point, the significance of it is still often overlooked, in spite of the fact that it is far from new[4]. … A point is not a diffeomorphism invariant entity, for diffeomorphisms move the points around. There are hence no observables of the form of the value of some field at a given point of a manifold, x.

[footnote 4:] "The original argument for the identification of the physical spacetime with a diffeomorphism equivalence class of metrics is due to Einstein and is called the hole argument."

(Smolin, 2001)

- But Einstein *refuted* hole argument with point-coincidence argument: points are *physically* individuated by what happens there (Hoefer, 1996; Maudlin, 2002).
- Hamiltonian GR was born in classical differential geometry.
- Classical differential geometry is naturally adapted to point-coincidence argument: Leibniz-friendly because contents (field values) are never separated from points.
- After Einstein 1915, "[f]or the next sixty five years, the Hole Argument was seen as a historical curiosity, little more than a misstep on the way to general relativity. . . . The basic thesis of the present note is that Einstein and the generations of physicists and mathematicians after him were right to reject the Hole Argument. (Weatherall, 2014)
- Modern geometry *primitively* (unphysically) individuates points, makes them separable from contents (active diffeomorphisms) like Samuel Clarke—at least mathematically.

- Taking physical equivalence classes is a belated effort to have Leibnizian physics with Clarkean absolutist mathematics.
- 1980s hole argument revival due to primitive individuation.
- Self-generated puzzle (Weatherall, 2014)?
- "...worry...that the diffeomorphism associated with the Hole Argument is meant to be a so-called 'active' diffeomorphism, whereas I am interpreting it as a 'passive' diffeomorphism.[21] This, I claim, is the only way in which $\tilde{\psi}$ can be interpreted–and, for that matter, how $\tilde{\phi}$ in the previous section should be interpreted as well." (Weatherall, 2014)
- For observations, use physically (not mathematically) individuated points p, needing 5 scalars to observe 1 scalar because 4 pick out p (Rovelli, 2002; Rovelli, 2006).
- Worry that *e.g.*, $R(p)$ isn't observable (Smolin, 2001; Earman, 2002) resolved by Einstein's point-coincidence argument: being p is bound up with what happens there.

Invariance of Observables $\int d^4x\sqrt{-g}$ and $\int d^4x\sqrt{-g}R(g)$?

... the spacetime volume is an observable for compact universes. So is the average over the spacetime, of any scalar function of the physical fields... [w]here the average is taken using the volume element defined by the spacetime metric. (Smolin, 2001)

Hamiltonian constraint observables. *These are observables which are constructed according to the rules of the hamiltonian formulation for systems with time reparameterization invariance. They must do at least one of the following things, i) have vanishing Poisson bracket with the classical hamiltonian constraint, ii) [reduced phase space...], iii) commute with the quantum hamiltonian constraint. (Smolin, 2001)*

- Drop all spatial dependence (toy theory), shift disappears.
- $\{\xi p, \sqrt{-g}\} = \{\xi p, N\sqrt{h}\} = -\xi\sqrt{h} \neq div.$
- $\{\epsilon\mathcal{H}_0, N\sqrt{h}\} = \epsilon N\pi/2 =$ (on-shell) $-\epsilon N\sqrt{h}K \neq div.$
- $\int dt\{\xi p + \epsilon\mathcal{H}_0, N\sqrt{h}\} =$ (on-shell) $-\xi\sqrt{h} - \epsilon\sqrt{h}h^{ij}\dot{h}_{ij}$
  $\neq BT$: 4-volume integral is *not* invariant under separate FCC.
- So set $\xi = \dot{\epsilon}$ and get $\int dt\{G, N\sqrt{h}\} = -\int dt(\epsilon\sqrt{h})_{,0}$:
  4-volume integral $\int d^4x\sqrt{-g}$ invariant if $\epsilon \rightarrow 0$ at time ends.
- $\mathcal{R} = \sqrt{-g}R(g) = N\sqrt{h}({}^3R + K_{ij}K^{ij} - K^2)+$
  $2(\sqrt{-g}n^\mu\nabla_\nu n^\nu - \sqrt{-g}n^\nu\nabla_\nu n^\mu)_{,\mu}$ (Wald, 1984, p. 464)
  $= N\sqrt{h}(K_{ij}K^{ij} - K^2) - 2\left(\frac{\sqrt{h}_{,0}}{N}\right)_{,0}$ (toy theory).
- Remove velocities *via* Hamilton's equations:
  $\mathcal{R}_H =_{df} N\mathcal{H}_0 + \frac{\partial(N\mathcal{H}_0)}{\partial h_{ab}}h_{ab} - \frac{\partial(N\mathcal{H}_0)}{\partial\pi^{ab}}\pi^{ab}.$
- $\{\xi p, \mathcal{R}_H\} = -\frac{\xi}{N}\mathcal{R}_H \neq divergence$: not gauge.

- First-class secondary $\mathcal{H}_0$: $\{\epsilon^{\perp}\mathcal{H}_0, \mathcal{R}_H\} = -\epsilon^{\perp}\left(\frac{\mathcal{R}_H}{N}\right)_{,0}$ on-shell $\neq divergence$: not gauge.
- $\{G, \mathcal{R}_H\} = \{\epsilon^{\perp}\mathcal{H}_0 + \dot{\epsilon}^{\perp}p, \mathcal{R}_H\} = -(N\xi^0\mathcal{R}_H/N)_{,0} = -\pounds_{\xi}(\sqrt{-g}R)$ on-shell.
- $\int d^4x\sqrt{-g}R(g)$ invariant under G but not each FC constraint.
- G does the job correctly—on-shell. Off-shell $\{G, \mathcal{R}_H\} = \epsilon^{\perp}\left[\dot{h}_{ab} - \frac{\partial H}{\partial \pi^{ab}}\right]\left[\frac{\partial^2\mathcal{H}_0}{\partial h_{ab}\partial h_{cd}}h_{cd} - \frac{\partial^2\mathcal{H}_0}{\partial h_{ab}\partial \pi^{cd}}\pi^{cd}\right] + \epsilon^{\perp}\left[\dot{\pi}^{ab} + \frac{\partial H}{\partial h_{ab}}\right]\left[\frac{\partial^2\mathcal{H}_0}{\partial \pi^{ab}\partial h_{cd}}h_{cd} + \frac{\partial^2\mathcal{H}_0}{\partial \pi^{ab}\partial \pi^{cd}}\pi^{cd}\right] - \left[\epsilon^{\perp}\frac{\mathcal{R}_H}{N}\right]_{,0}.$
- "You can't always get what you want, but if you try sometimes, you just might find—you get what you need!"
- Using G.

Loosened Definition of Observables

- Vectors, tensors, *etc.* translatable *via* tensor transformation rule, hence likewise observable in sense of covariance.
- Change found by $\pounds_\xi$ is real B-series change: different properties at different times (*c.f.* (Earman, 2002)).
- The new year starts in New York when the ball hits the bottom: temporal conventions also expressible.
- **In**variance under internal transformation: $\{O, G_{int}\} = \mathbf{0}$, and
- **Co**variance (translation rule) under external, $\{O, G_{ext}\} = -\pounds_\xi O \neq \mathbf{0}$.
- Early work pointed implicitly in that direction (Anderson and Bergmann, 1951).
- Meets Bergmann's demand of H-L equivalence (Bergmann and Komar, 1962; Bergmann, 1961; Bergmann, 1962).

- Novel Lagrangian-inequivalent postulates started to appear in Bergmann & Schiller (Bergmann and Schiller, 1953, section 4).
- Observers don't change powers as theorists choose formalisms, so Hamiltonian-Lagrangian equivalence is obligatory.
- Observables as determined by local observations (Bergmann, 1962, p. 250) (*c.f.* (Torre, 1993; Earman, 2002)).
- Typical definition (**each**, $\mathbf{0}$) violates both locality and H-L equivalence.
- $\{O, G\} = -\pounds_{\xi} O$ (on-shell) where O has special Lie derivative with group property possessed by geometric objects (Bergmann, 1949).

Four Definitions of Observables (*construed with physical, not primitive, point individuation)

Proponent	Generator	PB Effect	$\vec{E}$	$g_{\mu\nu}$	Vary	H-L equiv
Dirac, Bergmann 'lemma'	Each FCC	$\mathbf{0}$	No	No	No	No
Pons Salisbury Sundermeyer	Team G	$\mathbf{0}$	Yes	*	E-mag Yes GR *	E-mag Yes GR *
Kuchař	Each FCC but $\mathcal{H}_0$	$\mathbf{0}$ E-mag 0 $\mathcal{H}_i$ $\mathcal{H}_0$?	No	No	E-mag Yes GR time not space	No
JBP	Team G	$\mathbf{0}$ E-mag $\pounds_\xi$ GR	Yes	Yes	Yes	Yes

Einstein-Maxwell Observables and Legendre-Projectability

- Observables with external and internal gauge groups tricky?
- Remove velocities by mixing internal gauge transformation with external coordinate transformation—especially time (Pons et al., 2000).
- $G_R[\xi]$ generates pure Maxwell gauge transformation: $\{G_R[\xi], A_\mu\} = \partial_\mu \xi$.
- $G_V[\vec{\eta}]$ generates spatial coordinate change + certain Maxwell gauge change: $\{G_V[\vec{\eta}], A_\mu\} = -\eta^\alpha F_{\alpha\mu} \neq -\pounds_\eta A_\mu$, $\eta^\alpha n_\alpha = 0$.
- $G_S[\zeta^0]$ generates time coordinate change + Maxwell gauge change on-shell: $\{G_S[\zeta^0], A_\mu\} = -(\zeta^0 n^\alpha) F_{\alpha\mu} \neq -\pounds_{\zeta^0 n} A_\mu$.
- Effects on $F_{\alpha\mu}$: $\{G_R[\xi], F_{\alpha\mu}\} = \partial_\alpha \{G_R[\xi], A_\mu\} - \mu \leftrightarrow \alpha = 0$.

- $\{G_V[\vec{\eta}], F_{\alpha\mu}\} = \partial_\alpha \{G_V[\vec{\eta}], A_\mu\} - \mu \leftrightarrow \alpha = -\pounds_\eta F_{\alpha\mu}$.
- $\{G_S[\zeta^0], F_{\alpha\mu}\} = \partial_\alpha \{G_S[\zeta^0], A_\mu\} - \mu \leftrightarrow \alpha = -\pounds_{\zeta^0 n} F_{\alpha\mu}$.
- If observables need $\{G, O\} = 0$ for all gauge generators, then $F_{\mu\nu}$ is observable in Maxwell's theory but unobservable in Einstein-Maxwell.
- My definition (internally invariant, externally covariant) makes $F_{\mu\nu}$ observable in Einstein-Maxwell.
- $g_{\mu\nu}(p)$, $R(p)$, *etc.* also observable: Lie derivative (on-shell).
- Mixing internal with external for projectability affects A_μ but not $F_{\mu\nu}$.
- Supergravity—can it fit internal *vs.* external dichotomy?
- Real issue is ineffable *vs.* effable, implying invariance *vs.* covariance.
- *E.g.*, field GR's change of background metric (Grishchuk et al., 1984) is ineffable, so observables *in*variant under (external!) $\delta\eta_{\mu\nu} = \pounds_\xi \eta_{\mu\nu}$, $\delta g_{\mu\nu} = 0$, matter $\delta u = 0$.

Reduced Phase ~~Space~~ Space-time

- GR on reduced phase *space* has been sought (Belot and Earman, 2001) and criticized (Thébault, 2012).
- Phase *space* is too small for many-fingered time and velocity-dependent gauge transformations.
  Use phase space-*time* (Marmo et al., 1983; Sugano et al., 1986; Sugano et al., 1985; Lusanna, 1990; Rovelli, 1991).
- Hamiltonian change of time coordinate is only *on-shell* (Fradkin and Vilkovisky, 1977; Castellani, 1982; Pons et al., 2000; Pitts, 2014a), so usual notion of full reduction fails.
- Vacuum: 10 q's and 10 p's at each point + 1 time, 8 FC constraints, $20\infty^3 + 1$ dimensional phase space-time.
- Degree of freedom count: $(20\infty^3 - 2 \cdot 8\infty^3)/2 = 2\infty^3$.
- *C.f* dropping N, β^i primaries: $(12\infty^3 - 2 \cdot 4\infty^3)/2 = 2\infty^3$.

Conclusions and Conjectures

- Repaired Hamiltonian GR has change where there is Lagrangian change: absence of time-like Killing vector field.
- Hamiltonian trouble largely from Widespread Belief about FC constraints, partly from neglecting internal *vs.* external.
- Repaired Hamiltonian has locally varying observables: geometric objects and tensor calculus.
- Nuts-and-bolts, doing the math.
- Distinctly quantum issue of constraints remains (Pons, 2005).
- Quantum gravity conjectures: usual 'observables' too strict?
- Maybe quantum amplitudes merely externally *co*variant?

Anderson, J. L. (1958).
Reduction of primary constraints in generally covariant field theories.
Physical Review, 111:965–966.

Anderson, J. L. and Bergmann, P. G. (1951).
Constraints in covariant field theories.
Physical Review, 83:1018–1025.

Belot, G. and Earman, J. (2001).
Pre-Socratic quantum gravity.
In Callender, C. and Huggett, N., editors, *Philosophy Meets Physics at the Planck Scale*, pages 213–255. Cambridge University Press, Cambridge.

Bergmann, P. (1957).
Topics in the theory of general relativity.

In *Lectures in Theoretical Physics, Brandeis University Summer Institute in Theoretical Physics*. W. A. Benjamin, New York.
notes by Nicholas A. Wheeler.

Bergmann, P. G. (1949).
Non-linear field theories.
Physical Review, 75:680–685.

Bergmann, P. G. (1956).
Introduction of "true observables" into the quantum field equations.
Il Nuovo Cimento, 3:1177–1185.

Bergmann, P. G. (1961).
Observables in general relativity.
Reviews of Modern Physics, 33:510–514.

Bergmann, P. G. (1962).
The general theory of relativity.
In Flügge, S., editor, *Prinzipien der Elektrodynamik und Relativitätstheorie*, volume IV of *Handbuch der Physik*, pages 203–272. Springer, Berlin.

Bergmann, P. G. and Goldberg, I. (1955).
Dirac bracket transformations in phase space.
Physical Review, 98:531–538.

Bergmann, P. G. and Komar, A. (1962).
Observables and commutation relations.
In *Les Théories Relativistes de la Gravitation, Royaumont, 21-27 Juin 1959*, pages 309–325. Centre National de la Recherche Scientifique, Paris.

Bergmann, P. G. and Komar, A. B. (1960).

Poisson brackets between locally defined observables in general relativity.
Physical Review Letters, 4:432–433.

Bergmann, P. G. and Schiller, R. (1953).
Classical and quantum field theories in the Lagrangian formalism.
Physical Review, 89:4–16.

Brunetti, R., Fredenhagen, K., and Rejzner, K. (2015).
Quantum gravity from the point of view of locally covariant quantum field theory.
arxiv.org, 1306.1058v4.

Castellani, L. (1982).
Symmetries in constrained Hamiltonian systems.
Annals of Physics, 143:357–371.

Dirac, P. A. M. (1958).
The theory of gravitation in Hamiltonian form.
Proceedings of the Royal Society of London A, 246:333–343.

Dirac, P. A. M. (1964).
Lectures on Quantum Mechanics.
Belfer Graduate School of Science, Yeshiva University.
Dover reprint, Mineola, New York, 2001.

Earman, J. (2002).
Thoroughly modern McTaggart: Or, What McTaggart would have said if he had read the General Theory of Relativity.
Philosophers' Imprint, 2(3).
http://www.philosophersimprint.org/.

Fradkin, E. S. and Vilkovisky, G. A. (1977).
Quantization of relativistic systems with constraints:

Equivalence of canonical and covariant formalisms in quantum theory of gravitational field.
Ref.TH 2332.CERN.
http://cds.cern.ch/record/406087/.

Gèhèniau, J. and Debever, R. (1956).
Les quatorze invariants de courbure de l'espace riemannien à quatre dimensions.
Helvetica Physica Acta, 29:101–105.

Govaerts, J. (1991).
Hamiltonian Quantisation and Constrained Dynamics.
Leuven Notes in Mathematical and Theoretical Physics 4B.
Leuven University Press, Leuven.

Grishchuk, L. P., Petrov, A. N., and Popova, A. D. (1984).

Exact theory of the (Einstein) gravitational field in an arbitrary background space-time.
Communications in Mathematical Physics, 94:379–396.

Henneaux, M. and Teitelboim, C. (1992).
Quantization of Gauge Systems.
Princeton University, Princeton.

Hoefer, C. (1996).
The metaphysics of space-time substantivalism.
Journal of Philosophy, 93:5–27.

Kiefer, C. (2012).
Quantum Gravity.
Oxford University Press, 3rd edition.

Kingdom Magazine (n.d.).
Bo Jackson Knows Golf.

https://kingdom-mag.com/features/bo-jackson-knows-golf/.

Komar, A. (1955).
Degenerate scalar invariants and the groups of motion of a Riemann space.
Proceedings of the National Academy of Sciences of the United States of America, 41:758–762.

Komar, A. (1958).
Construction of a complete set of independent observables in the general theory of relativity.
Physical Review, 111:1182–1187.

Kuchař, K. V. (1986).
Canonical geometrodynamics and general covariance.
Foundations of Physics, 16:193–208.

Kuchař, K. V. (1993).

Canonical quantum gravity.
In Gleiser, R. J., Kozameh, C. N., and Moreschi, O. M., editors, *General Relativity and Gravitation 1992: Proceedings of the Thirteenth International Conference on General Relativity and Gravitation held at Cordoba, Argentina, 28 June–4 July 1992*, pages 119–150. Institute of Physics Publishing, Bristol.
gr-qc/9304012.

Lusanna, L. (1990).
An enlarged phase space for finite-dimensional constrained systems, unifying their Lagrangian, phase- and velocity-space descriptions.
Physics Reports, 185:1–54.

Marmo, G., Mukunda, N., and Samuel, J. (1983).

Dynamics and symmetry for constrained systems: A geometrical analysis.
Rivista del Nuovo Cimento, 6:1–62.

Maudlin, T. (2002).
Thoroughly muddled McTaggart: Or, How to abuse gauge freedom to generate metaphysical monstrosities.
Philosophers' Imprint, 2(4).
http://www.philosophersimprint.org/.

Mukunda, N. (1980).
Generators of symmetry transformations for constrained Hamiltonian systems.
Physica Scripta, 21:783–791.

Pitts, J. B. (2014a).
Change in Hamiltonian general relativity from the lack of a time-like Killing vector field.

Studies in History and Philosophy of Modern Physics, 47:68–89.
http://arxiv.org/abs/1406.2665.

Pitts, J. B. (2014b).
A first class constraint generates not a gauge transformation, but a bad physical change: The case of electromagnetism.
Annals of Physics, 351:382–406.
Philsci-archive.pitt.edu; arxiv.org/abs/1310.2756.

Pons, J., Salisbury, D. C., and Shepley, L. C. (1997).
Gauge transformations in the Lagrangian and Hamiltonian formalisms of generally covariant theories.
Physical Review D, 55:658–668.
gr-qc/9612037.

Pons, J. M. (2005).

On Dirac's incomplete analysis of gauge transformations.
Studies in History and Philosophy of Modern Physics, 36:491–518.
arXiv:physics/0409076v2.

Pons, J. M. and Salisbury, D. C. (2005).
Issue of time in generally covariant theories and the Komar-Bergmann approach to observables in general relativity.
Physical Review D, 71:124012.
gr-qc/0503013.

Pons, J. M., Salisbury, D. C., and Shepley, L. C. (2000).
Gauge transformations in Einstein-Yang-Mills theories.
Journal of Mathematical Physics, 41:5557–5571.
gr-qc/9912086.

Pons, J. M., Salisbury, D. C., and Sundermeyer, K. A. (2009).
Revisiting observables in generally covariant theories in the light of gauge fixing methods.
Physical Review D, 80:084015.
arXiv:0905.4564v2 [gr-qc].

Pons, J. M., Salisbury, D. C., and Sundermeyer, K. A. (2010).
Observables in classical canonical gravity: Folklore demystified.
Journal of Physics: Conference Series, 222:012018.
First Mediterranean Conference on Classical and Quantum Gravity (MCCQG 2009); arXiv:1001.2726v2 [gr-qc].

Pons, J. M. and Shepley, L. C. (1995).
Evolutionary laws, initial conditions, and gauge fixing in constrained systems.

Classical and Quantum Gravity, 12:1771–1790.
gr-qc/9508052.

Pons, J. M. and Shepley, L. C. (1998).
Dimensional reduction and gauge group reduction in Bianchi-type cosmology.
Physical Review D, 58:024001.
gr-qc/9805030.

Rosenfeld, L. (1930).
Zur Quantelung der Wellenfelder.
Annalen der Physik, 397:113–152.
Translation and commentary by Donald Salisbury, Max Planck Institute for the History of Science Preprint 381, http://www.mpiwg-berlin.mpg.de/en/resources/preprints.html, November 2009.

Rothe, H. J. and Rothe, K. D. (2010).
Classical and Quantum Dynamics of Constrained Hamiltonian Systems.
World Scientific, Hackensack, New Jersey.

Rovelli, C. (1991).
Time in quantum gravity: An hypothesis.
Physical Review D, 43:442–456.

Rovelli, C. (2002).
Partial observables.
Physical Review D, 65:124013.
gr-qc/0110035.

Rovelli, C. (2006).
A note on the foundation of relativistic mechanics. II: Covariant Hamiltonian general relativity.

In García-Compeán, H., Mielnik, B., Montesinos, M., and Przanowski, M., editors, *Topics in Mathematical Physics, General Relativity and Cosmology in Honor of Jerzy Plebański*, pages 397–407. World Scientific, Singapore. gr-qc/0202079.

Schouten, J. A. (1954).
Ricci-Calculus: An Introduction to Tensor Analysis and Its Geometrical Applications.
Springer, Berlin.

Shepley, L. C., Pons, J. M., and Salisbury, D. C. (2000).
Gauge transformations in general relativity—A report.
Turkish Journal of Physics, 24(3):445–452.
Regional Conference on Mathematical Physics IX, 9-14 Aug. 1999, Istanbul, Turkey.

Smolin, L. (2001).
The present moment in quantum cosmology: Challenges to the arguments for the elimination of time.
www.arxiv.org.
Slightly revised version of essay published in Robin Durie, *ed.*, *Time and the Instant*, Clinamen Press, Manchester (2000) pp. 112-143.

Sugano, R., Saito, Y., and Kimura, T. (1986).
Generator of gauge transformation in phase space and velocity phase space.
Progress of Theoretical Physics, 76:283–301.

Sugano, R., Sohkawa, T., and Kimura, T. (1985).
Relation between generators of gauge transformations and subsidiary conditions on state vectors. Point mechanical systems with arbitrary numbers of constraints.

Progress of Theoretical Physics, 73:1025–42.

Tambornino, J. (2012).
Relational observables in gravity: A review.
SIGMA, 8(017).
arXiv:1109.0740.

Thébault, K. P. Y. (2012).
Three denials of time in the interpretation of canonical gravity.
Studies in History and Philosophy of Modern Physics, 43:277–294.

Torre, C. G. (1993).
Gravitational observables and local symmetries.
Physical Review D, 48:R2373–R2376.

Wald, R. M. (1984).

General Relativity.
University of Chicago, Chicago.

Weatherall, J. O. (2014).
Regarding the "hole argument".
Forthcoming in The British Journal for the Philosophy of Science.
arXiv:1412.0303 [physics.hist-ph].

Wikipedia (4 June 2015).
Bo Jackson.
http://en.wikipedia.org/wiki/Bo_Jackson.