# The intrinsic Hamilton-Jacobi dynamics of general relativity and its implications for the semi-classical emergence of time

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Munich Center for Mathematical Philosophy, July 3-4, 2015

# Overview

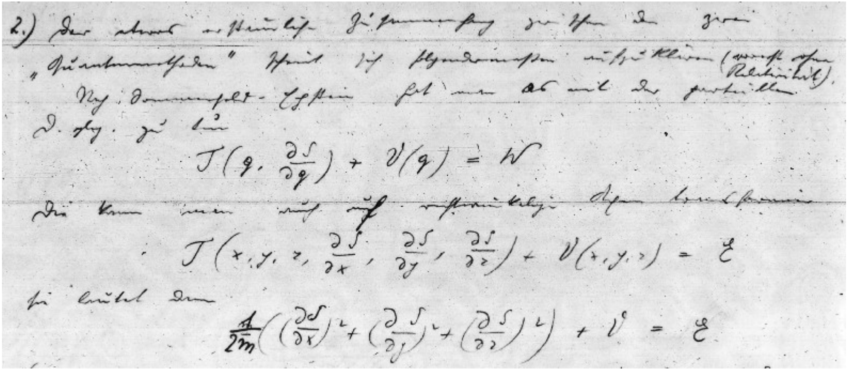
# 1. INTRODUCTION

## Questions to be addressed

- What is the status of four-dimensional diffeomorphism covariance in the Hamiltonian formulation of classical general relativity?
- What relationship do observables bear to this underlying symmetry?
- Is time a classical observable?
- What is the relation between the Wheeler-DeWitt equation, the Einstein-Hamilton-Jacobi equation from which it is derived, and general covariance?
- How can a 4-diffeomorphism invariant time emerge in a semi-classical Wheeler-DeWitt approach to quantum gravity?

# 2. A BRIEF HISTORY OF HAMILTON-JACOBI APPROACHES TO QUANTUM GRAVITY

# The old quantum theory as semiclassical quantum theory



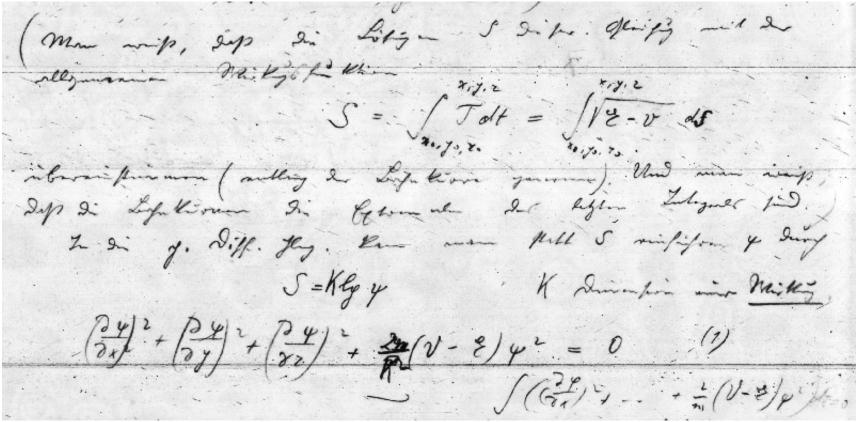
$$T\left(q, \frac{\partial S}{\partial q}\right) + V(q) = W$$

$$T\left(x, y, z, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}\right) + V(x, y, z) = E$$

$$\frac{1}{2m}\left(\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right) + V = E$$

Pages from Schrödinger's notebook.

# The old quantum theory as semiclassical quantum theory



$$S = \int_{x_0,y_0,z_0}^{x,y,z} T\,dt = \int_{x_0,y_0,z_0}^{x,y,z} \sqrt{\mathcal{E} - V}\,ds$$

$$S = K \lg \psi$$

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{2m}{K^2}(V - \mathcal{E})\psi^2 = 0 \quad (1)$$

Pages from Schrödinger's notebook.

# The role of Hamilton-Jacobi theory in anticipating structures of quantum theory

- The optical mechanical analogy has its roots in Hamilton's original introduction of his characteristic function in 1837. This is the same function that appears in the Sommerfeld-Epstein rule.
- The Hamilton principal function $S$ as a phase is consistent with the Schrödinger wave equation in the limit $\hbar \to 0$ The power series expansion of $S$ in $\hbar$ was first introduced independently by Wentzel and Brillouin in 1926. They in turn with Kramers in 1926 established the general conditions under which Sommerfeld-Epstein quantization agreed with wave mechanics. See Pauli's 1933 Handbuch der Physik article for an overview.

## The role of Hamilton-Jacobi theory in anticipating structures of quantum theory

- As also noted 1926, semi-classical wave packets that satisfy the correct classical equations of motion may be constructed through the superposition over complete principal function solutions of the Hamilton-Jacobi equation. These superpositions are of the form

  $$\int d\alpha e^{iS(x,t;\alpha)}$$

  The result follows as a consequence of the Hamiltonian dynamical equations
- These observations served as a point of departure of Peter Weiss's groundbreaking extension of the Hamilton-Jacobi formalism to field theory in 1936 [Weiss, 1936].

# Dirac in 1951 on the significance of Hamilton-Jacobi

From Dirac's 1951 foundational paper on constrained Hamiltonian dynamics, "The Hamiltonian form of field dynamics" [Dirac, 1951]

**1. Introduction.** In classical dynamics one has usually supposed that when one has solved the equations of motion one has done everything worth doing. However, with the further insight into general dynamical theory which has been provided by the discovery of quantum mechanics, one is lead to believe that this is not the case. It seems that there is some further work to be done, namely to group the solutions into families (each family corresponding to one principal function satisfying the Hamilton-Jacobi equation). The family does not have any importance from the point of view of Newtonian mechanics; but it is a family which corresponds to one state of motion in the quantum theory, so presumably the family has some deep significance in nature, not yet properly understood.

# Dirac in 1951 on the significance of Hamilton-Jacobi

The importance of the family is brought out by the Schrödinger form of quantum mechanics and not by the Heisenberg form. The latter is in direct analogy with the classical Hamiltonian equations of motion and does not require any grouping of the solutions. The Schrödinger form goes beyond this in ascribing importance to the concept of a quantum state, subject to the principle of superposition and described by a solution of Schrödinger's wave equation, and this concept requires the introduction of families of solutions for its analogue in classical mechanics, the Schrödinger equation itself being the analogue of the Hamilton-Jacobi equation.

# The standard approach to semiclassical canonical quantization of gravity

Let us now look more closely at the promise and at the limitations of the standard approach to a semi-classical canonical quantization of general relativity via the Wheeler-DeWitt equation. It was inspired by a so-called Einstein-Hamilton-Jacobi equation first written down by Peres in 1962 [Peres, 1962].

IL NUOVO CIMENTO  VOL. XXVI, N. 1  1° Ottobre 1962

**On Cauchy's Problem in General Relativity – II.**

A. PERES (*) (**)

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# Einstein-Hamilton-Jacobi equation

Peres replaced the canonical momenta $p^{ab}$ in the Hamiltonian constraint

$$\mathcal{H}_0 = -\sqrt{g}\,{}^3R + \frac{1}{\sqrt{g}}\left(p^{ab}p_{ab} - \frac{1}{2}p^2\right) = 0,$$

with a functional derivative of a Hamilton principal function $S$ with respect to the spatial metric field $g_{ab}$. It is important to note that this replacement was not derived by Peres from a variation of the gravitational action.

This equation inspired Bryce DeWitt. In his own words, from his 1999 paper "The Quantum and Gravity: The Wheeler-DeWitt equation": [DeWitt, 1999]

# Origins of the Wheeler-DeWitt Equation according to DeWitt

*"John Wheeler, the perpetuum mobile of physics, called me one day in the early sixties. I was then at the University of North Carolina in Chapel Hill, and he told me that he would be at the Raleigh-Durham airport for two hours between planes. He asked if I could meet with him there and spend a while talking quantum gravity. John was pestering everybody at the time with the question: What are the properties of the quantum mechanical state functional Psi and what is its domain? He had fixed in his mind that the domain must be the space of 3-geometries, and he was seeking a dynamical law for Psi."*

# Origins of the Wheeler-DeWitt Equation according to DeWitt

*"I had recently read a paper by Asher Peres which cast Einstein's theory into Hamilton-Jacobi form, the Hamilton-Jacobi function being a functional of 3-geometries. It was not difficult to follow the path already blazed by Schrödinger, and write down a corresponding wave equation. This I showed to Wheeler, as well as an inner product based on the Wronskian for the functional differential wave operator. Wheeler got tremendously excited at this and began to lecture about it on every occasion."*

# Origins of the Wheeler-DeWitt Equation according to DeWitt

*"I wrote a paper on it in 1965, which didn't get published until 1967 because my Air Force grant was terminated and the Physical Review in those days was holding up publication of papers whose authors couldn't pay the page charges. My heart wasn't really in it [...] But I thought I should at least point out a number of intriguing features of the functional differential equation, to which no one had yet begun to devote much attention: [...] The fact that the wave functional is a wave function of the universe and therefore cannot be understood except within the framework of a many-worlds view of quantum mechanics [...] In the long run one has no option but let the formalism provide its own interpretation. And in the process of discovering this interpretation one learns that time and probability are both phenomenological concepts."*

# 3. LESSONS OF THE RELATIVISTIC FREE PARTICLE MODEL

# The free particle action

Consider the reparamterization covariant free particle action

$$S = -\int (-\dot{q}^2)^{1/2} d\theta = \int L d\theta.$$

I review a technique for employing this action to construct a classical and quantum model that establishes a dynamical correlation between observable variables, recognizing that the parameter $\theta$ is itself not observable. Our task is to relate this parameter to a measurable physical quantity.

# The free particle action increment

The standard point of departure is to is to consider two independent changes to slightly different solutions of the equations of motion. The first independent variation is characterized by the fact that the new solutions have the same value of configuration variables at a slightly increased final evolution parameter as the original solutions had at the original final evolution time. The second variation simply alters the configuration variables at at the original final evolution time. I put some stress on this procedure since it appears not to be well appreciated that this procedure can be carried out also for singular systems, as in this model.

# The free particle action increment

The result in this case is

$$dS = \frac{\partial L}{\partial \dot{q}^\mu} dq^\mu - \left( \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L \right) d\theta =: \tilde{p}_\mu dq^\mu - \tilde{H}(\dot{q}) d\theta.$$

We notice of course that this is a singular system; the momenta $\tilde{p}_\mu(\dot{q}) := \frac{\partial L}{\partial \dot{q}^\mu} = \dot{q}_\mu(-\dot{q}^2)^{-1/2}$ are not independent. In fact, since the Lagrangian is homogeneous of degree one in the velocities, $\tilde{H} := \frac{\partial L}{\partial \dot{q}^\mu} \dot{q}^\mu - L \equiv 0$. The constraint takes the form $\tilde{p}^2 + 1 \equiv 0$, and we have $\tilde{H} = (-\dot{q}^2)^{1/2} \left( \tilde{p}^2 + 1 \right)$.

# A free particle intrinsic time

We now ask how we can gain information from $dS$ on the measurable physical evolution of the single particle system. In this case the answer is clear. We could simply choose the reparametrization scalar $q^0$ as the evolution time. In doing so we establish a relation between the in principle measureable spatial position of the particle and the measurable Minkowski time $q^0$. This is a choice of intrinsic time - intrinsic in the sense that the evolution parameter itself is measurable.

As we shall see, there is in general a two step procedure for doing this.

# Step 1 of intrinsic coordinate fixation procedure

In this particular case the first step is already accomplished since $q^0$ is already a configuration space variable. But in order to stress the fact that the isolation of the $q^0(\theta)$ does not automatically imply that one has made an intrinsic coordinate choice I point out that this variable does undergo variations under the reparameterizations of the form $\theta' = \theta - (-\dot{q}^2)^{-1/2}\xi(\theta)$. The phase space generator of these variations is $G(\xi) = \frac{\xi}{2}H$.

# "Hamilton-Jacobi" equation does not fix an intrinsic parameter

Since we have not yet made a parameter choice we have a phase space constraint $H = 0$. Then it is natural to ask what would be the consequence of representing the momenta in the constraint as $p_\mu = \frac{\partial S}{\partial q^\mu}$ and interpreting the constraint as a differential equation to be satisfied by $S$. In other words, look for solutions of

$$\eta^{\mu\nu}\frac{\partial S}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} + 1 = 0.$$

It is significant for us that solutions of this equation do not give us directly solutions $q^\mu(\theta)$ of the reparameterization covariant Euler-Lagrange equations. Additional information is required.

# "Hamilton-Jacobi" equation does not fix an intrinsic parameter

Given a solution $S$, we can set

$$p_\mu = \frac{\partial S}{\partial q^\mu}, \tag{1}$$

Then we appeal to the Hamiltonian equation $\dot{q}^\mu = \lambda(\theta) p^\mu$ where we pick the function $\lambda$, obtaining the first order differential equation $\dot{q}^\mu = \frac{\partial S}{\partial q^\mu}$ which can then be integrated. The point is that only when this function has been selected have a made a choice of gauge. In other words, the "Hamilton-Jacobi equation" continues within this formalism to be a constraint, and we have simply managed to solve the constraint. This is the reason that I have enclosed the expression in quotation marks. It is not a true Hamilton-Jacobi equation.

# Step 2 of intrinsic coordinate fixation procedure

Of course, what really motivates our interest in this example is the means that is available to find $q^a$ as a function of $q^0$. In other words, we want to make an explicit intrinsic parameter choice. And there is a natural way of doing this using the "Hamilton-Jacobi equation". It does give us directly $S$ as a function of $q^a$, $q^0$ and of three independent constants $\alpha^a$. And solutions for $q^a$ as a function of $q^0$ can be obtained in the usual manner in Hamilton-Jacobi theory by taking derivatives $\frac{\partial S}{\partial \alpha^a}$. Thus the "Hamilton-Jacobi equation" brings with it a natural choice of intrinsic parameter - due to the fact that this natural choice is one of the configuration variables!

## Step 2 of intrinsic coordinate fixation procedure

It is also possible to make the intrinsic parameter choice directly in the action increment $dS$. We simply interpret $q^0$ as the evolution parameter and the momenta as phase space variables - subject of course to the constraint. Thus we can solve for $p_0 = -(p^a p_a + 1)^{1/2} := H_{intrinsic}$, so that the increment in the action in intrinsic coordinates is

$$dS = -H_{intrinsic} dq^0 + p_a dq^a.$$

From this expression we deduce the true intrinsic Hamilton-Jacobi equation

$$\frac{\partial S}{\partial q^0} + \left( \frac{\partial S}{\partial q^a} \frac{\partial S}{\partial q_a} + 1 \right)^{1/2} = 0.$$

# Reparameterization invariant variable

Finally, although it is obvious that the intrinsic dynamics does not depend on the parameter $\theta$, and is thus invariant under reparameterizations, it is instructive to see how this intrinsic choice yields variables expressed in terms of the $q^\mu(\theta)$ in an arbitrary parameterization but which are invariant under reparameterizations. These invariant variables are (See [Pons *et al.* , 2009b])

$$\mathcal{I}_{q^a} = q^a(\theta) + \frac{p^a}{p^0} q^0(\theta) - \frac{p^a}{p^0}\theta.$$

The coefficients of each power of $\theta$ are invariant under the active transformations generated by $G(\xi)$. This means that when we go to the quantum theory and we consider wave functions $\psi(\mathcal{I}_{q^a}, \theta)$, these wave functions will satisfy $\hat{H}\psi(\mathcal{I}_{q^a}, \theta) = 0$.

# 4. INTRINSIC COORDINATES AND GAUGE FIXING IN GENERAL RELATIVITY

# The realization of the full diffeomorphism group in phase space

A misunderstanding of the role of constraints has led to many of the errors in both the formulation and in the interpretation of the Wheeler-DeWitt equation. We will show that the appropriate phase space formalism that will admit the incorporation of the quantum of action and retain the full four-dimensional diffeomorphism symmetry is a formalism that retains the lapse and shift as configuration variables. This will in turn yield a fruitful fully covariant semi-classical approach to quantum gravity. (See [Pons *et al.* , 1997] [Pons & Salisbury, 2005] [Pons *et al.* , 2009a] )

# The diffeomorphism-induced canonical transformation group

- Global translations in time (time evolution) are not realizeable in general relativity as a canonical phase space transformations. This is commonly known - though not fully appreciated - as the decomposition of infinitesimal diffeomorphims into hypersurface tangential and perpendicular transformations
$$\delta x^\mu = \delta^\mu_a \xi^a + n^\mu \xi^0.$$
- The notion of "multi-fingered" time was introduced by Kuchar in 1972 [Kuchař, 1972] before it was understood that the full 4-diffeomorphism-induced group could be realized as a canonical transformation group.

# The diffeomorphism-induced canonical transformation group

Global rigid translation in time is generated in a fixed gauge by the Rosenfeld-Bergmann-Dirac Hamiltonian (also known as the ADM Hamiltonian)

$$H_{RBD} = \int d^3x \left( N^\mu \mathcal{H}_\mu + \lambda^\mu \pi_\mu \right),$$

where the $\lambda^\mu$ are spacetime functions, related via the Hamiltonian equations of motion to time rates of change of the lapse and shift,

$$\frac{\partial N^\mu}{\partial t} = \lambda^\mu.$$

# The diffeomorphism-induced canonical transformation group

General infinitesimal diffeomorphism-induced transformations of the full 4-metric and conjugate momenta are generated by

$$G_\xi(t) = \int d^3x \left(P_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + N^\rho C^\nu_{\mu\rho} P_\nu)\xi^\mu\right).$$

Taking into account the time-dependence of this generator, a standard calculation demonstrates that even though the spacetime functions $\lambda^\mu$ in the $H_{RBD}$ Hamiltonian are not dependent on the phase space variables, the formalism yields the correct variation of these functions under an arbitrary infinitesimal four-dimensional diffeomorphism. In other words, the phase space formalism (retaining lapse and shift as canonical phase space variables) is fully covariant under arbitrary time coordinate foliations.

# The diffeomorphism-induced canonical transformation group

Thus the Hamiltonian and true Hamilton-Jacobi formalism is covariant under the full four-dimensional diffeomorphism group.

Most importantly, the group can be employed to construct variables that are invariant under the action of the 4-D diffeomorphism group. The framework within which we work may appear at first sight to be contrary to the spirit of general relativity. We assume that spacetime coordinates are given and we will describe how a physical significance can be assigned to these coordinates, namely by relating them to measurable spacetime curvature.

# The diffeomorphism-induced canonical transformation group

The diffeomorphism-induced group acts actively on the phase space variables that are taken to be functions of these fixed spacetime coordinates. Similarly to particle model, our task is to identify functionals of these variables that can serve as spacetime landmarks, i.e., intrinsic coordinates. Once this choice is made we will have succeeded in establishing univocal correlations between these landmarks and remaining phase space variables. These correlations are invariant under the action of the group, just as the relation between $q^0$ and $q^a$ is insensitive to the active action of the reparameterization group.

# Intrinsic coordinates must be spacetime scalars

Intrinsic coordinates must be spacetime scalars as one can see with this simple argument : We suppose that the coordinates $x^\mu$ have been fixed by the condition that $x^\mu = X^\mu(g(x), p(x))$. We investigate how this relation transforms under a change of coordinates $x'^\mu = f^\mu(x)$. Then we must have

$$x'^\mu = f^\mu(x) = f^\mu\left(X(g(x), p(x)\right) = f^\mu\left(X(g'(x'), p(x))\right),$$

We deduce that since $f$ is arbitrary,

$$X^\mu(g(x), p(x)) = X^\mu(g'(x'), p'(x')). \tag{2}$$

In other words, the $X^\mu$ must be spacetime scalar functions.

# Intrinsic coordinates must be spacetime scalars

It has been shown that if the scalar condition is satisfied then no physical solutions are eliminated, and if is not satisfied then the fixation of coordinates is not unique [Pons *et al.* , 2010].

# Scalar spacetime curvature candidates

In general relativity with material sources we have at our disposal at least fourteen scalars that can be constructed from the Riemann curvature tensor. They generally involve quadratic or cubic powers of this tensor.

# Invariants constructed using intrinsic coordinates

It has also been shown how to construct the observables that correspond to given intrinsic coordinate choices $x^\mu = X^\mu[g_{ab}, p^{cd}]$ [Pons *et al.* , 2009b]. The outcome is that there corresponds to every phase space variable (including the lapse and shift) a series expansion in powers of the intrinsic coordinates, the coefficients of which are invariant functionals of the metric. These coefficients are invariant in the conventional sense that variations of the metric under changes of the coordinate parameters does not change their values.

# 5. THE INTRINSIC HAMILTON-JACOBI APPROACH TO GENERAL RELATIVITY

## Gravitational action increment

We begin by writing the increment in the gravitational action as

$$dS_{GR} = \int d^3x \left(\tilde{p}^{ab} dg_{ab} + \tilde{P}_\mu dN^\mu\right) - \int d^4x N^\mu \mathcal{H}_\mu,$$

where the tilde signifies that the momenta are to be conceived as configuration-velocity functions. This formula is obtained by assuming that at a fixed final time one computes the action for a slightly different new solution of Einstein's equations where one varies the final time and assumes that the values of the metric components at the new final time coincide with the values at the original end time in some chosen system of coordinates. Of course when so conceived the $\tilde{P}_\mu$ vanish identically. And we must satisfy the secondary constraints $\mathcal{H}_\mu = 0$.

# Implementation of intrinsic coordinates

Since spacetime scalars necessarily depend on time derivatives of the metric, it is not immediately obvious how one can identity them in the nonvanishing contribution to the action increment.

Fortunately, Karel Kuchař's pioneering work from 1973 [Kuchar, 1973] suggests a way to proceed - via the appropriate canonical phase space transformation. Our language and method is however distinct. We speak not of "embeddings", nor "bubble time" or "multifingered time", but rather simply canonical phase space transformations leading to a dynamics that is still covariant under the 4-D diffeomorphism group.

# Implementation of intrinsic coordinates

Thus we seek a canonical transformation such that

$$S_{GR} = \int d^3x \left(p^{ab} dg_{ab} + P_\mu dN^\mu\right) - \int d^4x N^\mu \mathcal{H}_\mu \left[g_{ab}, p^{cd}\right]$$

$$= \int d^3x \left(p^I dg_I + p^A dg_A + \frac{\delta G}{\delta g_I} dg_I + \frac{\delta G}{\delta X^\mu} dX^\mu + P_\mu dN^\mu\right)$$

$$- \int d^4x N^\mu \mathcal{H}'_\mu \left[g_A, p^B, X^\mu, \pi_\mu\right].$$

Here we let $g_A$ represent two of the components of $g_{ab}$, with $g_I$ representing the remaining four and similarly for the conjugate momenta, while the $\pi_\mu$ are the momenta conjugate to $X^\mu$.

# Step 1 accomplished

This is the first step - which was not necessary in the particle model.

# Einstein-Hamilton-Jacobi equations

The Einstein-Hamilton-Jacobi equation - first written down by Peres in 1962 [Peres, 1962] - is the Hamiltonian constraint

$$\mathcal{H}_0\left[g_{ab}, \frac{\delta S}{\delta g_{cd}}\right] = 0.$$

We now have countless new Einstein-Hamilton-Jacobi equations (corresponding to different choices of intrinsic coordinates $X^\mu$:

$$\mathcal{H}_0\left[g_A, X^\mu, \frac{\delta S}{\delta g_B}, \frac{\delta S}{\delta X^\nu}\right] = 0.$$

But none of these relations give a fixed coordinate choice!

# Intrinsic Hamiltonians

There is however a "natural" coordinate choice for each choice of $X^\mu$, just as $q^0$ was a natural choice of $\theta$ in the particle model. We let $x^\mu = X^\mu$. Finally, in order to respect the constraints, we solve the constraints and the intrinsic coordinate conditions $x^\mu = X^\mu\left(g_{ab}, p^{cd}\right)$ for $g_I$ and $p^J$ and substitute this expressions into $\pi_I$ whereby the $\pi_I$ become explicit functionals of $g_A, p^B$ and $x^\mu$. Then since there is no incremental change in the $X^a$ in this gauge, the non-vanishing contribution to the action becomes the intrinsic canonical one-form

$$dS_{intrinsic} = \int d^3x \left(p^A dg_A + \pi_0\left[g_A, p^B, x^\mu\right] dx^0\right).$$

# Intrinsic Hamiltonian and intrinsic Hamilton-Jacobi equation

Thus we deduce that the intrinsic Hamiltonian is

$$H_{intrinsic} := -\pi_0 \left[ g_A, p^B, x^\mu \right],$$

with a corresponding intrinsic Hamilton-Jacobi equation,

$$\frac{\partial \bar{S}}{\partial t} + H_{intrinsic} \left[ g_A, \frac{\delta \bar{S}}{\delta g_B}, x^\mu \right] = 0.$$

## Intrinsic coordinate transformations

We now have the liberty to undertake an arbitrary finite transformation to new intrinsic coordinates $X'^{\mu} = f^{\mu}(X)$. This is merely a point transformation, and can therefore be realized in phase space as a canonical transformation.

We require that

$$\int d^3x\, X^{\mu} d\pi_{\mu} = -\int d^3x \pi'_{\mu} dX'^{\mu} + \int d^3x dF\left[X', \pi\right].$$

# Intrinsic coordinate transformations

The generator is

$$F = \int d^3x \, f^{-1\mu}(X')\pi_\mu,$$

The transformed intrinsic Hamiltonian is

$$H'_{intrinsic} = -\pi'_0 = \frac{\delta F}{\delta X'^0}.$$

Thus once we have found one admissible set of intrinsic coordinates, we now have a means of canonically transforming to all possibe intrinsic coordinates. The freedom of choice is the original diffeomorphism freedom!

# 6. A COSMOLOGICAL EXAMPLE

# Robertson Walker cosmology with massless scalar source

We illustrate these ideas with a simple cosmological example, an isotropically expanding universe with vanishing curvature, vanishing cosmological constant, and a massless scalar source field $\phi$. The line element is

$$ds^2 = -N(t)^2 dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

with Lagrangian

$$L = \frac{1}{2N}\left(-\frac{6}{\kappa}a\dot{a}^2 + a^3\dot{\Phi}^2\right),$$

where $\kappa := 8\pi G$.

# Intrinsic time candidate

We find that the quadratic Riemann scalar $R^1 := R_{\alpha\alpha'\beta\beta'} g^{\beta\beta'\gamma\gamma'} R_{\gamma\gamma'\delta\delta'} g^{\delta\delta'\alpha\alpha'}$, where $g^{\beta\beta'\gamma\gamma'} := 2 g^{\beta[\gamma} g^{\gamma']\beta'}$, simplifies to a power of $a^2 p_a^{-1}$ for this highly symmetric solution.

(Bergmann and Komar showed in 1960 that all curvature scalars can be expressed in terms of the 3-metric and conjugate momenta [Bergmann & Komar, 1960] .)

# York time

We therefore take $T = \frac{1}{3}\kappa a^{-2} p_a$ as one of our new phase space variables. With this choice $T$ will range from $-\infty$ to 0. Note that since $\dot{a} = -\frac{\kappa N}{6a} p_a$, this variable is actually minus twice the Hubble parameter $N^{-1} a^{-1} \dot{a}$.

It is important to note that this York time - equivalently the extrinsic curvature 3-diffeo scalar - is a spacetime scalar only under arbitrary infinitesimal coordinate transformations. It will be important to learn what form this time assumes when spatial inhomogeneities are introduced in the cosmological model.

# Conjugate momentum

The momentum conjugate to $T$ is $p_T = -\frac{1}{\kappa}a^3$. The constraint in terms of the new variables is

$$H' = \frac{3}{4}p_T T^2 - \frac{p_\phi^2}{2\kappa p_T} = 0.$$

This leads to a new Einstein-Hamilton-Jacobi equation

$$\frac{3}{4}\left(\frac{\partial S}{\partial T}\right)^2 T^2 - \frac{p_\phi^2}{2\kappa} = 0.$$

This relation does not fix $\phi$ as a function of $t$. The full reparameterization freedom in $t$ remains.

# Intrinsic Hamiltonian

But there is now a natural choice. $T$ is a new configuration variable analogous to $q^0$ in the particle model. Let us take $t = T$ and eliminate $p_T$ by solving the constraint. This delivers an intrinsic Hamiltonian

$$H_{intrinsic} = -p_T = -\frac{1}{(3\kappa/2)^{1/2}} \frac{p_\phi}{(-t)}$$

with Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{(3\kappa/2)^{1/2}} \frac{1}{t} \frac{\partial S}{\partial \phi} = 0.$$

The complete solution is

$$S_i(\phi, t; \alpha) = e^{-\alpha\phi_0 - \frac{\alpha}{(3\kappa/2)^{1/2}} \ln(-t) + \alpha\phi}$$

where $\alpha$ and $\phi_0$ are constants.

# Intrinsic Hamiltonian and semi-classical physics

One obtains the general classical solution for $\phi$ in the usual manner by setting $0 = \frac{\partial S_i}{\partial \alpha}$. This insures that in passing to the quantum theory we obtain a wave packet that follows the classical trajectory by forming an appropriate superposition

$$\Psi(\phi, t) = \int d\alpha f(\alpha) e^{S_i(\phi, t; \alpha)/\hbar}.$$

This wave function satisfies the Schrödinger equation

$$H_i\left(t, -i\hbar\frac{\partial}{\partial\phi}\right)\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

as a consequence of the intrinsic Hamilton-Jacobi equation.

# Analogue Wheeler-DeWitt approach

What if we decided to employ the analogue Wheeler-DeWitt equation, using the transformed constraint, i.e.

$$\frac{3}{4} p_T^2 T^2 - \frac{p_\phi^2}{2\kappa} = 0 \rightarrow \left( -\frac{3}{4} t^2 \frac{\partial^2}{\partial t^2} + \frac{1}{2\kappa} \frac{\partial^2}{\partial \phi^2} \right) \Psi(\phi, t) = 0.$$

rather than the intrinsic Schrödinger equation?

Then it turns out that if solutions are assumed of the form $\Psi = e^{iS/\hbar}$, then one can show after considerable labor, after expanding $S$ in powers of $\kappa$, that $S$ satisfies the intrinsic Hamilton-Jacobi equation. The lesson to be drawn is that the Schrödinger equation is far more efficient.

# Invariant $\phi$

The explicit invariant $\phi$ is obtained by employing the finite reparameterization group. It is a power series in $t$ of nested Poisson brackets. Only the coefficient of $t^0$ delivers a nontrivial invariant, namely $a(t)p_a(t)$. This is indeed a constant - independent of $t$.

# Canonical change of intrinsic coordinate

The following canonical transformation transforms from York time to proper time:

$$T' = -T^{-1} = f(T).$$

Therefore according to our general prescription we can perform a point canonical transformation to obtain the corresponding new intrinsic Hamiltonian,

$$p'_{T'} = p_T \frac{df^{-1}(T')}{dT'} = \frac{p_\phi}{(3\kappa/2)^{1/2}} T' \frac{d(-T')^{-1}}{dT'} = \frac{p_\phi}{(3\kappa/2)^{1/2}} \frac{1}{T'}.$$

Note that this time ranges from 0 to $\infty$.

## All coordinates can be expressed in terms of spacetime scalar functionals!

But we need not stop here. We can actually imitate any choice of time coordinate by chosing new intrinsic times as $\lambda(T')$, for arbitrary positive definite functions $\lambda$.

This is actually a substantiation of a general result. For every choice of coordinates in general relativity there corresponds a choice of intrinsic coordinates.

Thus we have in effect turned the relativity principle on its head. For every choice of coordinates there corresponds a curvature-based prescription of spacetime landmarks. Evolution in terms of these landmarks is unique. Thus each and every spacetime coordinate choice yields a physically distinguishable dynamics.

# 7. INTRINSIC GENERAL RELATIVISTIC SCHRÖDINGER EQUATIONS

# Schrödinger equation

The natural next step to take, in complete analogy with Schrödinger's original quantization procedure, is to write down an intrinsic Schrödinger equation for each choice of intrinsic coordinates:

$$H_{intrinsic}\left[g_A, -i\hbar\frac{\delta}{\delta g_B}; x^\mu\right]\Psi\left[g_A, x^\nu\right] = i\hbar\frac{\partial}{\partial t}\Psi\left[g_A, x^\nu\right].$$

We are assured that we can construct the correct semiclassical

limit, for each of the choices for $X^\mu$, from solutions of the form $\Psi = \sigma e^{iS/\hbar}$ where $S$ is a complete solution of the intrinsic Hamilton-Jacobi equation.

# 8. CONCLUSIONS

# Diffeomorphism group and new phase space variables

- The full four-dimensional diffeomorphism-induced group is realizeable in phase space as a canonical transformation group
- Spacetime curvature based phase space variables can be found that can serve as intrinsic temporal clocks and spatial rods.
- The freedom in selecting these spacetime landmarks corresponds to the original diffeomorphism freedom.
- The dynamics in terms of the new variables is fully covariant under the 4-D diffeomorphism-induced group.

# Einstein-Hamilton-Jacobi and Wheeler-DeWitt

- For each choice of new phase variables there exists a corresponding Einstein-Hamilton-Jacobi equation

$$\mathcal{H}_0\left[g_A, X^\mu, \frac{\delta S}{\delta g_B}, \frac{\delta S}{\delta X^\nu}\right] = 0.$$

- $S$ does not yield classical solutions of Einstein's equations as functions of intrinsic coordinates
- The corresponding Wheeler-DeWitt equation

$$\mathcal{H}_0\left[g_A, X^\mu, -i\hbar\frac{\delta}{\delta g_B}, -i\hbar\frac{\delta}{\delta X^\nu}\right]\Psi[g_A, X^\mu] = 0,$$

does implement the intrinsic coordinate choice at the semi-classical level.

# Intrinsic Schrödinger equation

- But the intrinsic Schrödinger equation

$$H_{intrinsic}\left[g_A, -i\hbar\frac{\delta}{\delta g_B}; x^\mu\right]\Psi\left[g_A, x^\nu\right] = i\hbar\frac{\partial}{\partial t}\Psi\left[g_A, x^\nu\right],$$

is technically and conceptually simpler.

# The multiplicity in intrinsic coordinate challenge

- We have a classical canonical transformation scheme for obtaining the infinitude of possible intrinsic Hamiltonians.
- Each yields a unique classical evolution.
- The quantum mechanical challenge is to construct a theory in which all of these in general unitarily inequivalent diffeomorphism invariant evolutions are taken into account. A full description of reality appears to require the collective use of all possible intrinsic times.

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