Dispensing with the continuum, 25 years later

Sam Sanders

Remembering Patrick Suppes, LMU, Sept. 9, 2015

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I was impressed by his program *Dispensing with the continuum*, esp. its grand aims. I will discuss the program's history and my contributions towards the completion of its radical version.

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constructive $\approx$ effective $\approx$ computable $\approx$ implemented on a PC.

First steps: The Chuaqui-Suppes system

Rolando Chuaqui and Patrick Suppes introduced the logical system $\mathfrak{CS}$ in the late eighties, as part of the *Dispensing* program.

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We also have, from (5), if $dt > 0$

$$P_n(t) = P_n(t-dt) + \lambda dt(P_{n-1}(t-dt) - P_n(t-dt)) + \sum_{i=0}^{n-2} \lambda_i dt(P_i(t-dt) - P_n(t-dt)),$$

that is

$$P_n(t-dt) = P_n(t) - (\lambda(P_{n-1}(t-dt) - P_n(t-dt))\,dt + \sum_{i=0}^{n-2} \lambda_i(P_i(t-dt) - P_n(t-dt)\,dt).$$

As above, by (7) and (8)

$$\left|\sum_{i=0}^{n-2} \lambda_i(P_i(t-dt) - P_n(t-dt))\right| \leq (n-2)\gamma\,dt,$$

where the right side is an infinitesimal. So

$$P_n(t-dt) \approx P_n(t) - \lambda(P_{n-1}(t-dt) - P_n(t-dt))\,dt \quad (dt).$$

Let $dt < 0$. Changing dt by $-dt$ in the formula above, we get

(10). $$P_n(t+dt) \approx P_n(t) + \lambda(P_{n-1}(t+dt) - P_n(t+dt))\,dt \quad (dt)$$

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A major component of physics is that predictions made by theories are compared to experimental data (using computers nowadays).

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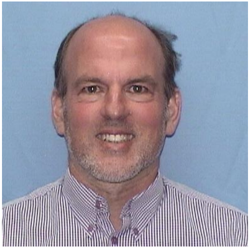
Elementary Recursive Nonstandard Analysis (ERNA) is a streamlined version of $\mathfrak{CS}$ introduced by Rick Sommer and Suppes.



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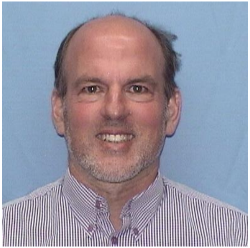

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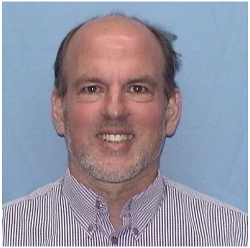

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In other words, ERNA's isomorphism theorem is an attempt to solve the 'computability' problem of $\mathfrak{CS}$.

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Corollary (First fundamental theorem of Calculus)

Let f be continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$. Then F is S-differentiable on $[a, b]$ and $\frac{d}{dx}F(y) \approx f(y)$ holds for all $a \ll y \ll b$.

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These kind of conditions are the price to pay for working over $^*\mathbb{Q}$ the hyperrational numbers, rather than the reals $\mathbb{R}$.

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Theorem (Dirac delta theorem)

Let g be near-standard and S-continuous on $[a, b]$ and let π be a hyperfine partition of $[a, b]$. If $\frac{|\pi|}{\varepsilon} \approx 0$ and $a \ll 0 \ll b$ and $\delta_\varepsilon(x)$ is a delta function, then

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Question: How to remove infinitesimals from ERNA-theorems?

ERNA's isomorphism theorem I

Purpose: Replace all infinitesimals in an ERNA-theorem by very small (but in $\mathbb{Q}$) numbers, while keeping the theorem correct.

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The isomorphism theorem of Chuaqui-Sommer-Suppes:

Theorem 6.1. Isomorphism Theorem *Let $\mathcal{M}$ be a reasonably sound model of nonstandard analysis, and let $\mathcal{T}$ be a finite set of terms in the language of* ERNA, *closed under subterms. Then there are arbitrarily large (truly) finite natural numbers b, such that there is an isomorphism f from $\mathcal{T}^{\mathcal{M}} =_{\text{def}} \{\tau^{\mathcal{M}} : \tau \in \mathcal{T}\}$, the $\mathcal{M}$ interpretations of the elements of $\mathcal{T}$, to a finite subset of the (truly) standard rationals that satisfies the following:*

FINITE MODELS OF ELEMENTARY RECURSIVE NONSTANDARD ANALYSIS 91

1. $f(g(\tau_1, \dots, \tau_l)) = g(f(\tau_1), \dots, f(\tau_l))$, *where g is any function symbol of the language other than ν_0 and ϵ_0.*
2. $f(\nu_0) = n_0$ *and* $f(\epsilon_0) = 1/n_0$, *for some* $n_0 \geq b$.
3. $\mathrm{Inf}(\tau)$ *holds in $\mathcal{M}$ if and only if* $|f(\tau)| \leq \frac{1}{b}$.
4. $\mathcal{N}(\tau)$ *holds in $\mathcal{M}$ if and only if $f(\tau)$ is a natural number.*
5. $\tau \leq \sigma$ *holds in $\mathcal{M}$ if and only if* $f(\tau) \leq f(\sigma)$.

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ERNA's isomorphism theorem II

Theorem (Isomorphism theorem II, S.)

Let $\mathcal{T}$ be a finite set of closed intensional terms in the language of ERNA, *not including* min *and closed under subterms. There is a bijection f from $\mathcal{T}$ to a finite set of rationals such that*

1. $f(0) = 0$, $f(1) = 1$ *and* $f(\omega) = n_0$, *for some* $n_0 \in \mathbb{N}$,
2. $f(g(\tau_1, \dots, \tau_k)) = g(f(\tau_1), \dots, f(\tau_k))$, *for all non-atomic terms in* $\mathcal{T}$,
3. $\tau \approx 0$ *iff* $|f(\tau)| < \frac{1}{b}$, *for some* $n_0 > b \in \mathbb{N}$,
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But how to remove infinitesimals from mathematical results, so that the latter may be compared to experimental data?

The unreasonable effectiveness of NSA

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The system $\mathfrak{P}$ is introduced in:

van den Berg, Briseid, Safarik, A functional interpretation of nonstandard arithmetic, APAL2012

The following results are taken from:

Sanders, The unreasonable effectiveness of Nonstandard Analysis, arXiv2015.

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But (1) is the notion of continuity (with a modulus t) used in constructive and computable analysis (Bishop, Weihrauch, etc).

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But (2) is the theorem expressing continuity implies Riemann integration from constructive and computable analysis.

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GRAND CLAIM: Every theorem of constructive/computable mathematics is the result of TERM EXTRACTION applied to a 'precursor' theorem from pure NSA.

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Any questions?