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Structuralism and Mathematical Practice in Felix Klein's Work on Non-Euclidean Geometry

Francesca Biagioli

Foundations of Mathematical Structuralism
MCMP, LMU Munich
12–14 October 2016

Outline

1. Mathematical structuralism in the development of Klein's thought
2. From projective metrical geometry to the group-theoretical view of geometry
3. Structuralist themes in Klein's epistemological writings
4. Alternative philosophical views in the reception of Klein: Bertrand Russell and Ernst Cassirer
5. Concluding remarks on Klein's foundation of mathematical structuralism

Felix Klein and mathematical structuralism

- Klein played a decisive role in the development of the abstract concept of group (Wussing 1969).
- Riemann and Klein on the concept of manifold (Torretti 1978).
- Suppes (2002) refers to Klein to move from intuitive ideas to more formal ideas of invariance and symmetry.
- Category theory as a generalization of Klein's Erlangen Program (Eilenberg and Mac Lane 1945; cf. Marquis 2009).
- Fortune (and misfortune) of the "Erlanger Programm"
  - Birkhoff and Bennett (1988); cf. Hawkins (1984), Rowe (1989).

Mathematical structuralism in the development of Klein's thought I

Klein, Felix. 1871a. "Notiz, betreffend den Zusammenhang der Liniengeometrie mit der Mechanik starrer Körper." *Mathematische Annalen* 4.

___. 1871b. "Über die sogenannte Nicht-Euklidische Geometrie." *Mathematische Annalen* 4, 573-625. Part 2, 1873.

___. 1872. "Vergleichende Betrachtungen über neuere geometrische Forschungen." Erlangen: Deichert. Repr. *Mathematische Annalen* 43 (1893)... Or the "Erlangen Program."

Lie, Sophus. 1888-1893. *Theorie der Transformationsgruppen*, 3 Bde, Leipzig.

Klein, Felix.1890. "Zur Nicht-Euklidischen Geometrie." *Mathematische Annalen* 37.

___. 1910. "Über die geometrischen Grundlagen der Lorentzgruppe." *Physikalische Zeitschrift* 12 (1911): 17-27.

Mathematical structuralism in the development of Klein's thought II

In the same years [1890s] I was able to cultivate my interests in mechanics and mathematical physics, as I have been intending to do since the beginning of my studies in mathematics. The first physical investigations in the theory of relativity emerged a few years later, and rapidly attracted general attention. I suddenly recognized that my classification of 1872 included even these investigations and provided the simplest way to clarify the newest physical (or even philosophical) ideas from a mathematical viewpoint. It was not long before I decided to elaborate on my idea, first in my lecture on the Lorentz group from 1910… (Klein, *Gesammelte mathematische Abhandlungen*, 1921, p.413)

Mathematical structuralism in the development of Klein’s thought III

- In the course of Klein’s work metrical projective geometry provided:

1. A foundation of metrical geometry via Klein’s model of non-Euclidean geometry
2. A useful method to clarify the relations between geometry, number theory, and the theory of functions
3. A conceptual or, as Klein put it, “rational” foundation of special relativity (Klein, 1910, p.21).

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Klein's sources

- Synthetic approach to projective geometry: Jakob Steiner, Christian von Staudt
- Analytic approach: August Ferdinand Möbius, Julius Plücker, Alfred Clebsch
- The theory of algebraic invariants: George Boole, James Joseph Sylvester, Arthur Cayley
- The transformation of the concept of space from a necessary presupposition to an object of research: Gauss, Riemann, Dedekind

A gap in von Staudt's treatment of projective geometry: Dedekind's axiom of continuity

Analytic geometers did not pay much attention to von Staudt's researches. This may have been because of the widespread idea that the essential aspect of von Staudt's geometry lies not so much in the projective approach as in the synthetic form. Von Staudt's considerations have a gap, which can only be filled by using an axiom, as described later on in the text. The same gap affects the extension of the method of von Staudt, as intended here. But our considerations concern not so much the extension as the original domain. The problem can be solved by specifying the analytical content of von Staudt's considerations, regardless of purely spatial ideas. Such a content can be summarized by demanding that projective space be represented by a numerical three-fold extended manifold. Besides, this is an assumption which lies at the foundation of any speculation about space. (Klein 1873, p.132, note)

Klein's "geometrical" interpretation of distance and the classification of geometries I

In order to determine the distance between two given points, I imagine them to be connected by a straight line. This line intersects the fundamental surface in two other points so that a cross-ratio of four points is constructed. I call the logarithm of this cross-ratio multiplied by a constant *c* the distance between the given points. (Klein 1871, p. 574)

- Euclidean and non-Euclidean metrics
  - Elliptic: Imaginary second-order surface.
  - Hyperbolic: The inner points of a real non-degenerate surface of second order.
  - Parabolic: The circle at infinity; one point is taken twice.

Klein's definition of distance and the classification of geometries II

- Hesse's principle of transfer: Suppose a manifold A has been investigated with reference to a group, and by any transformation A is converted into A', then B becomes B' and the B'-based treatment of A' can be derived from the B-based treatment of A (Klein 1893, p. 72).
- Klein uses this principle to prove the equivalence of elliptic, hyperbolic, and parabolic geometries with the three cases of manifolds of constant curvature (i.e., >, <, and = 0, respectively) according to Beltrami's "Teoria fondamentale degli spazi a curvatura costante" (1869).

From metrical projective geometry to the group-theoretical view of geometry

As long as our geometrical investigations are based on one and the same transformation group, the geometric content remains unvaried (Klein 1893, p.73).

- Given a manifold and a group of transformations of it; to develop the theory of invariants relating to that group (in the 1921 edition of the Erlangen Program Klein refers here to his papers on non-Euclidean geometry)
- Geometrical properties redefined as relative invariants of transformation groups.

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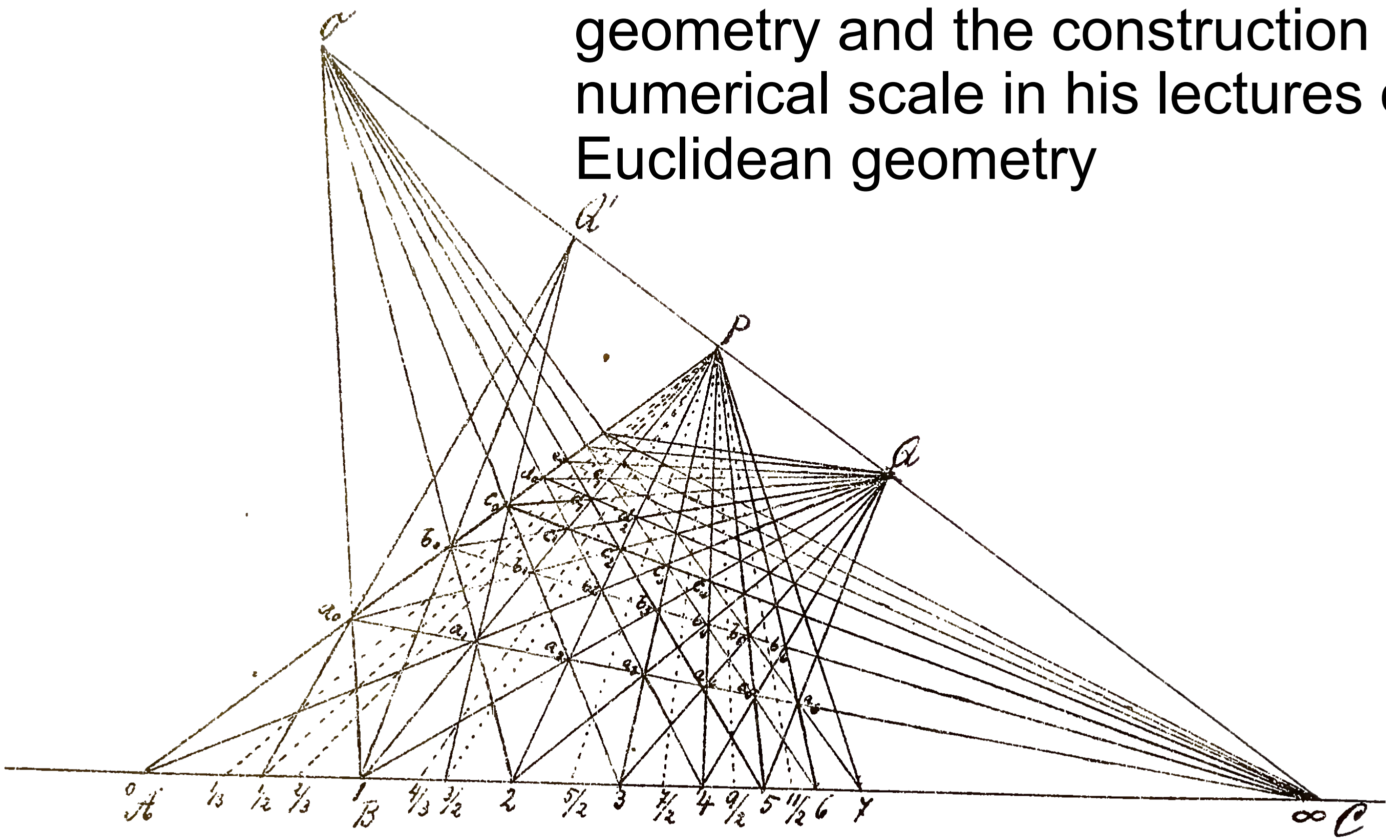
Klein's lectures on non-Euclidean geometry (1889-90) I

- Empiricism vs. Idealism
  - Hermann von Helmholtz, Moritz Pasch
  - Klein's distinction between pure and applied mathematics or mathematics of precision (calculation with real numbers) and mathematics of approximation (calculation with approximate values):

  Logical rigor [Dedekind's structuralism!] is required in the foundation of mathematical theories; that which is given in intuition and experiment is approximate and subject to revision.

  It is not so much a matter of inferring correct conclusions from correct premises, as to obtain those conclusions which follow with foreseeable correctness from approximately correct premises, or also to say to what extent further inferences can be made (p. 314).

Friedrich Schur or Klein-Lüroth-Zeuthen and Darboux? Klein’s prove of the fundamental theorem of projective geometry and the construction of a numerical scale in his lectures on non-Euclidean geometry



Klein, “Zur Nicht-Euklidischen Geometrie” (1890) and the Clifford-Klein problem of space form

I consider axioms to be the postulates by which we read exact assertions into inexact intuition. Thereby, I can clarify my stance towards the theory of irrationals. Certainly, the construction of irrational numbers was triggered by the seeming continuity of spatial intuition. Since I do not attribute any precision to spatial intuition, however, I will not want the existence of irrationals to be derived from such an intuition. I think that the theory of irrationals should be developed and delimited arithmetically, to be then transferred to geometry by means of axioms, and hereby enable the precision that is required for the mathematical consideration. (p.572)

Klein’s lectures on non-Euclidean geometry (1889-90) II

Let us assume that the space about us exhibits a Euclidean or a hyperbolic structure. We can by no means infer from this that space has an infinite extent. Because, for instance, Euclidean geometry is entirely compatible with the hypothesis that space is finite, a fact that has been formerly overlooked. The possibility of ascribing a finite content to space whatever its geometrical structure, is particularly welcome because the idea of an infinite expanse, which was originally looked upon as a substantial progress of the human mind, gives rise to many difficulties, e.g. in connection with the problem of mass distribution (Klein [1889-90]1928, p. 270; Engl. trans. in Torretti 1978, p. 152).

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Russell, *An Essay on the Foundations of Geometry* (1897)

Since these systems are all obtained from a Euclidean plane, by a mere alteration in the definition of distance, Cayley and Klein tend to regard the whole question as one, not of the nature of space, but of the definition of distance. Since this definition, on their view, is perfectly arbitrary, the philosophical problem vanishes – Euclidean space is left in undisputed possession, and the only problem remaining is one of convention and mathematical convenience. (p.30)

- Distance, as a relation between two points, is part of our basic knowledge about space (p.36).
- Projective coordinates as the number of houses in a street

Cassirer vs. Russell (and Paul Natorp) in *Substanzbegriff und Funktionsbegriff* (1910)

The geometrical axioms are not copies of the real relations of sense perception, but they are postulates by which we read exact assertions into inexact intuition. (p. 103)

As in the case of number we start from an original unit from which, by a certain generating relation, the totality of the members is evolved in fixed order, so here we first postulate a plurality of points and a certain relation of position between them, and in this beginning a principle is discovered from the various applications of which issue the to totality of possible spatial constructions. (p.88)

Cassirer's later remarks on Klein's structuralism I

Here again it is unnecessary to introduce the unreal elements as individuals leading some sort of mysterious existence side by side with the real points; the only logically and mathematically meaningful statement that can be made about them refers to the permanence of the relations that are embodied and expressed in them. But of course the symbolic thinking of mathematics does not content itself simply with apprehending these relations in abstracto; it demands and creates a special sign for the logical and mathematical relationship that is present in them and ultimately treats the sign itself as a fully valid, legitimate, mathematical object. (Cassirer 1929/1957, *The Philosophy of Symbolic Forms*, vol. 3, p.397)

Cassirer's later remarks on Klein's structuralism II

In his historical survey of the development of mathematical thought during the nineteenth century Felix Klein declared that one of the most characteristic features of this development is the progressive "arithmetization" of mathematics. Also in the history of modern physics we can follow this process of arithmetization. From Hamilton's quaternions up to the different systems of quantum mechanics we find more and more complex systems of algebraic symbolism. The scientist acts upon the principle that even in the most complicated cases he will eventually succeed in finding an adequate symbolism which will allow him to describe his observations in a universal and generally understandable language. (Cassirer, *An Essay on Man*, 1944, p.276)

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Concluding Remarks on Klein's Foundation of Mathematical Structuralism I

- Klein and Dedekind's logical structuralism (cf. Reck 2003)
- Set-theoretic structuralism
  - Sets as primitive
  - The issue of extendibility
- Ante rem structuralism
  - Platonism
- Structuralism in category theory
  - A systematic classification of geometries
  - The "formal supervenience" of group-theoretical properties upon geometric properties

Concluding Remarks on Klein's Foundation of Mathematical Structuralism II

Klein was well aware of the alternative: it is possible to put the transformation group first and define a geometry afterwards *or* start from a geometry given in one way or another and then consider its transformation group. In the latter case, the relationship between geometric properties and transformation groups can be put in terms of a relationship which has attracted the attention of philosophers over the last twenty-five years, although in fields which have nothing to do with mathematics and logic. Indeed, I claim that one can say that group-theoretical properties *formally supervene* upon geometric properties. (Marquis 2009, p.36)

- Cf. "Abstraction" (and "creation") in neo-Kantianism
- Mathematical practice and heuristics?

Thank you for your attention

francesca.biagioli@uni-konstanz.de

This research was kindly supported by the EU FP7 Marie Curie Zukunftskolleg Incoming Fellowship Programme, University of Konstanz (grant no. 291784).