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THE STRUCTURALIST ROOTS OF MATHEMATICAL UNDERSTANDING. EARLY FRENCH STRUCTURALISM RECONSIDERED

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Foundations of Mathematical Structuralism

Plan

1. Introduction
2. Lautman (Bourbaki) *reconsidered*
3. Poincaré *reconsidered* (after Lautman and Wittgenstein): structure as a 'mixed-universel'
4. Conclusion

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1° General Philosophical Approach

In the last decades, many studies have aimed to avoid classical philosophical positions in mathematical ontology and epistemology as platonism, nominalism, formalism, but also structuralism, by looking at mathematical practice as source for finding either new perspectives to the problems the classical positions discuss or to discuss until now neglected questions.

Naturally, to do mathematics cannot solve problems of philosophy of mathematics. So my general question is this:

Can the practical turn in philosophy of mathematics produce some progress on problems belonging to pragmatism as a philosophical (metaphysical) method, and in return, what can pragmatism offer to the practical turn in philosophy of mathematics? Or, more restricted to the subject of this conference: can a pragmatic-practical position shed some lights on structuralism in mathematics?

2° My general thesis about the function of structures

I propose a reconsideration of mathematical 'structuralism' as it was designed by Albert Lautman and Henri Poincaré. Their positions, while being very different, can be considered as a source of ideas which, from the point of view of a rational reconstruction, seems to suggest a modern sounding contribution to the solution of the philosophical difficulties of contemporary non-eliminative structuralism:

> For mathematical understanding, structures as special universals are a necessary tool, i. e. the condition of the possibility to understand mathematical practice.

In this paper I hope to convince you that this is a plausible hypothesis. I cannot justify it in its generality.

The main problem of modern structuralism

The ontological status of places and the problem of the identity of the structure itself are dependent — as Linnebo formulated in his talk — on the interpretation of the relation '*one over many*' between a structure and systems. For example, if one defines by abstraction a structure by fundamental properties that apply to all objects that instantiate the structure: what is the status of these objects? And if a structure simply is a universal that is predicated of each of the system, one has the problem with self instantiation.

3) Outlook on my interpretation of the main problem

In the following, I will interpret structures as 'universals' but not as resorting to a supersensible world; they are archetypes, characterized in a threefold way:

(a) they are *functions* (and not inborn contents) of our intellect, i.e.

(b) tools for the creation of models suggested by stipulated systems that

(c) exemplify in Goodman's sense the structures.

This position can be read as an emanation of Poincaré's and Lautman's philosophical ideas.

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Albert Lautman

(This chapter is a short version of a paper written together with Jean Petitot, forthcoming)

Between 1930 and 1940 three brilliant Ph.D. students and friends at the *École Normale Supérieure* of Paris were amongst those who militantly introduced, following their comrades creating the Bourbaki group, Hilbert's axiomatics into the French context. Jacques Herbrand (1908-1931), Jean Cavaillès (1903-1944), and Albert Lautman (1908-1944). They had alas the common fate to disappear prematurely.

Lautman had Herbrand as daily instructor in mathematics. Together, they became friends and intellectual companions first with Claude Chevalley, then with Charles Ehresmann, both founders of the Bourbaki group.

Lautman's works have been reviewed and commented by personalities as important as Paul Bernays, John Barkley Rosser, Max Black, Ferdinand Gonseth, or André Lichnerowicz. He was shot in 1944 by the Nazis as a Jew and as a resistance fighter politically on the left

Albert Lautman's philosophical goal

Lautman wanted to synthetize four issues:

(1) the logical deductions in the sense of proof theory,

(2) the complicated, entangled and ubiquitous relations of unification allowed by the axiomatic-structural perspective,

(3) the psychology of the creative mathematician (with his personal vision, inspiration, inventiveness, artistic genius, style),

(4) the historical dynamic of theories.

Albert Lautman's mathematical samples

The spectrum of Lautman's mathematical interests concerns

Analytic functions

The uniformization theorem of compact Riemann surfaces (i.e. algebraic projective curves over C) and the theory of universal covering: any compact Riemann surface is a quotient of the Riemann sphere (genus $g = 0$), or of the complex plane C (genus $g = 1$) or of the Poincaré hyperbolic half-plane (genus $g > 1$). The theory of Abelian integrals and the Riemann-Roch theorem. The link between the topological structure of the surface and the dimension over C of the C-vector space of Abelian integrals of first kind (equivalence between the two definitions of the genus).

Number theory and Galois theory

Algebraic extensions of the field Q of rational numbers and class field theory (Hilbert, Furtwängler, Takagi, Artin, Hasse, Herbrand, Chevalley,Weil). Analytic theory of numbers: distribution of primes, zeta functions of number fields, theta functions.

Logic and metamathematic

Albert Lautman, *Mathematics, Ideas and the Physical Real,* London/New York: Continuum International Publishing Group, 2011

According to **Albert Lautman**, Hilbert

- "replaced the method of genetic definitions with that of axiomatic definitions, and, far from claiming to reconstruct the whole of mathematics from logic, introduced on the contrary, by passing from logic to arithmetic and from arithmetic to analysis, new variables and new axioms which extend each time the domain of consequences."

Lautman is THE philosopher who aimed at justifying philosophically the Bourbakist conception. Bourbaki inaugurated an axiomatic-structural point of view that could seemingly work without the need of metamathematics in Hilbert's sense.

Lautman's "organic" structure

Lautman emphasizes the "organic" structure of theories, a biological metaphor often used by the up-and-coming Bourbaki group:

- "The logicists of the Vienna Circle always assert their full agreement with Hilbert's school. Nothing is however more debatable. (...) The object studied is not the set of propositions derived from axioms, but the organized, structured, complete entities, having an anatomy and physiology of their own. (...) The point of view that prevails here is that of the synthesis of necessary conditions and not that of the analysis of primitive notions."

Structures: the mathematical purpose

Lautman wanted to clarify the classical difference between "two kinds of mathematics". His purpose, inherited from Hermann Weyl, was to solve the conflicts echoed in the mathematical practice through, on the one hand,

the *structural axiomatic method* used in Algebra where groups, fields, etc. are given as wholes, totalities and global domains and not through an explicit construction of individual elements and,

on the other hand, *the constructive method*, conceiving the real numbers and the operations of Analysis as constructions generalizing number theory. The tool he imagined is an *adequate* interpretation of the structural method so that the conflict in fact disappears in favor of the algebraic method where "the priority of the notion of domain can be asserted with respect to the notion of number."

Structures: the philosophical purpose reconsidered

The hypothetico-deductive bourbakist foundations were explicitly designed as neutral with respect to philosophical foundations but might be engaged in agreement with the philosophical interest in scientific practices renewed today: foundations as structural systematization.

In fact, if the contemporary practical turn can be seen as positioned in a field of tension between pragmatism and the working scientist, it was already the case in thirties.

What unites the pragmatist and the working mathematicien (Bourbaki) is that they refuse the hypostasis of mathematical objects either from a philosophical perspective or from the point of view of mathematical practice.

The 'working mathematician' Henri Cartan, one of the founders of Bourbaki, wrote in 1943:

- "The mathematician does not need a metaphysical definition; he must only know the precise rules to which are subject the use he has in mind [...] But who decides upon the rules?"

Bourbaki: the problem of the nature of beings' or of 'mathematical objects' is not addressed

Cartan sounds Wittgensteinian but is not so in reality: according to Cartan, the first mathematical reasoning on a certain area intuitively obey certain rules and if difficulties arise, the use of reasoning is adapted, etc. Thus, a mathematical reality is created through practice. What is the criterion of practice and rules that result that reality? In his historical notice of set theory, Bourbaki

- "recognized that the 'nature' of mathematical objects is ultimately of secondary importance, and that it matters little, for example, whether a result is presented as a theorem of a 'pure' geometry or as a theorem of algebra *via* analytical [Cartesian] geometry. In other words, the essence of mathematics [...] appeared as the study of *relations* between objects which do not of themselves intrude on our consciousness, but are known to us by means of *some* of their properties, namely those which serve as the axiom at the basis of their theory".

Lautman's stratified mathematical real

Lautman stratified the mathematical real into four layers:

- facts, entities, theories, and Ideas.

In the autonomous and historical movement of the elaboration of its theories, mathematics realizes so called dialectical Ideas but, contrary to the Hegelian dialectic, the structural schemas realize relations between *complementary*, and not necessarily contradictory, notions: local/global, intrinsic/extrinsic, essence/existence, continuous/discontinuous, finite/infinite, Algebra/Analysis, etc. Alongside facts, beings, and mathematical theories, they constitute the fourth layer.

Lautman's third layer

For Lautman, the relevant philosophy of mathematics begins only at the third layer because mathematical properties ('objects') or facts are of a structural kind that means theory dependent. Lautman gave the example of the property of divisibility of the number 21. If the domain is the field Q of rational numbers, the irreducible divisors are 3, 7; if the domain is obtained by the adjunction of $\sqrt{-5}$ to the field Q, the irreducible divisors are 3, 7 or $(1+2\sqrt{-5})$, $(1-2\sqrt{-5})$. It follows that

- "the ***problem*** of mathematical reality is posed neither at the level of facts, nor at that of entities, but at that of theories."

Theories are not only sets of formulas deducible from axioms but also (see above) organized wholes "having an anatomy and physiology of their own".

Lautman's metaphysics

To situate the relevant philosophy of mathematics at the *third* layer (the level of theories) corresponds perfectly to the sensibility of the mathematicians Lautman wanted to promote metaphysically. But what has been difficult to understand for these mathematicians (his friends), and *a fortiori* for logicians, has been the introduction of the *fourth* layer of dialectic Ideas:

- "At this level [the level of theories], the nature of the real divides into two (…) one that focuses on the unique movement of this theory, the other on the connection of Ideas that are incarnated in this movement."

Mathematics and Ideas

That what interests us today in Lautman is not his platonic solution as such, i.e. the proposal that the intrinsic reality of mathematical entities, facts or theories lies in their dialectical participation in Ideas which dominate them and which are themselves realities.

What is subtle is his insight in the essential difference between the nature of mathematical models and the structure. Ideas are not

- "the models whose mathematical entities would merely be copies, but […] the structural schemas according to which the effective theories are organized." [Lautman 2011, 199].

He is right by emphasizing that a structure is what mathematical models have in common and that this cannot be understood just by doing more mathematics.

Status of Dialectical Ideas

It is essential to understand that, according to Lautman, Ideas of dialectical connections have no ontological commitments and no anteriority with respect to their instantiations in theories, but rather raise questions and "are only the problematic relative to the possible situations of entities." In short, dialectic Ideas are historical driving forces and by no means irreducible essences of an intellectual world. They have to do with cultural evolution and no evolution is teleological. Nothing could be more erroneous than believing that Lautman attributed to Ideas some finality anticipating mathematical discoveries.

The level to which Ideas belong is a rather peculiar metamathematical level. At the time of Lautman, it was purely conceptual and not formal. But, as every meta-level, it raises the traditional question of an infinite regress.

Circle eviction

If we try treating structures as individuals and describing their relations, we risk treating structures themselves as positions in a theory (structure) of structures. Lautman avoids such a circle in the following way that refers in particular to a work of Julius Stenzel (1923) concerning philosophical technicalities on historical Platonism:

- "Metamathematics which is incarnated in the generation of ideas and numbers does not give rise in turn to a meta-metamathematics. The regression stops as soon as the mind has reached the schemas according to which the dialectic is constituted."

Lautman’s dialectical solution

In other words, Lautman’s point of view on structures cannot be substituted, without precautions, by a meta-structural point of view on structures.

Indeed, this limitation has a negative bearing on structuralism only if mathematical reality is defined as structural reality, what – as we have seen on the beginning of this paper – is not Lautman’s thesis. He is not even searching for a foundational program, but rather hoping to achieve a deeper understanding of mathematical practice. His solution resembles to a dialectical *ante rem* – *in re* version: philosophically, the concept of a structure is ‘dominated’ by dialectical Ideas that bring two perspectives together: (a) the structure as an essence of a form, realized in a specific mathematical matter, created by the form, and (b) the essence of a mathematical matter giving rise to the Ideas as forms.

Lautman and category theory

It is well known that Bourbaki developed his project in the context of set theory (axiomatic systems and their models) but that a better context would have been category theory, and that in fact many of Bourbaki's constructions are already preparing categorical concepts (projective and inductive limits, functors, etc.). But when Lautman worked out his philosophy (1933-1939) any formalization of this high level categorical layer of mathematical reality was completely lacking.

It is therefore natural to wonder if Lautman's fourth layer of mathematical reality could have something to do with category theory.

Fernando Zalamea Albert Lautman and the Creative Dialectic of Modern Mathematics. Lautman 2011, xxiii-xxxvii

As was emphasized by Fernando Zalamea

- "It is remarkable that Lautman's conceptions are able to take shape fully (that is, theorematically, with their corresponding 'procession of precisions') through the fundamental concepts of category theory. (...) Most of the structural schemas and schemas of genesis studied by Lautman in his main thesis can be explained and, above all, extended, through the aide of category theory."

But to articulate category theory with an evolutionary theory of resolution of antinomies, foundational aporias or themata is another story.

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Poincaré: Geometry

Poincaré holds the 'structural' view that we have no pre-axiomatic understanding of geometric primitives, that rigor demands that we eliminate appeals to intuition with respect to metrics in geometry, and that pure (metric) geometry is neither true nor false: it is the result of conventions. He argues that what science

- "can attain is not the things themselves, as the naive dogmatists think, but only the relations between the things; apart from these relations, there is no knowable reality".

Poincaré's epistemological 'relationism'

With respect to relations, Poincaré seems to be an epistemological platoniste.

Two questions arise:

a) How conceive relations without relata? (= 'relationalism')
b) What are the links between Poincaré's epistemological 'relationalism' and the geometrical conventions?

For the rest of this paper, I argue for the thesis that his 'relationalism' and his conventionalism are in fact two different aspects of his structural approach, which attenuates the problem of *ante rem* structuralism.

Poincaré and Hilbert

Whereas

- for Hilbert the expressions in geometric systems are by construction *schematic* axioms (which are neither true nor false),
- for Poincaré the expressions in geometric systems are by construction apparent hypotheses, which are neither true nor false, too.

According to Hilbert, the mathematical formalism requires a "finite" metamathematics in order to demonstrate the consistency of formal mathematical systems,

According to Poincaré, it is necessary to explain certain hypotheses with respect to a metamathematical standard and to decline the variants of these hypotheses in different sciences.

Both approaches have their own difficulties: very general and well known by Gödel's theorems for Hilbert, much less known for Poincaré: for our general purpose, they can here be neglected.

Poincaré: On the Foundations of Geometry. *The Monist* IX (1898)

According to Poincaré, the set of relations that hold between the geometric primitives constitute the form, not the matter, of geometric objects, and the form is what is studied in geometry:

- "What we call geometry is nothing but the study of formal properties of a certain continuous group; so we may say, space is a group".

The link between Poincaré's ontological 'relationalism' and his geometric conventions becomes now visible by recalling the first levels of the construction of geometric space.

Poincaré's construction of geometric space I

At the start, our body plays the role of a coordinate system with respect to which we locate an object in space. Locating an object in the representative space means for Poincaré reflecting on the sequence of actions needed to reach this object, that is, reflecting on sequences of muscular, and not spatial, sensations.

To classify these sensations, Poincaré introduces the essentially vague category of representative space: there is neither measure nor the possibility of speaking of constant axes with respect to our body, but thanks to it, we can compare sensations of the same kind and observe the proximity of two objects.

Poincaré's construction of geometric space II

In itself, all the sensations are different, since they are accompanied, for example, by "various olfactory or auditory sensations". Their indistinguishability is a consequence of our abstractive classification. We maintain that the representative space is not formed by a classification starting from motor sensations, but on the contrary that it is the necessary condition for a classification of motor sensations. It is a form of our understanding and not of our sensibility, since an individual sensation can exist without it.

Poincaré’s construction of geometric space III

The construction of geometric space now proceeds from the observable fact that a set of impressions can be modified at least in two distinct ways: on the one hand without our feeling muscular sensations, and on the other, by a motor action accompanied by muscular sensations. In the first case, Poincaré speaks of an external change, in the second, of an internal change.

First step

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Similarly to Carnap's *Aufbau*, the starting point (guided by experience) is for Poincaré the definition of two two-place relations satisfying certain *minimal* empirical conditions: an *external chance a* (with '*x a y*' for '*x changes in y* ***without*** *muscular sensation*') and an *internal change S* (with '*x S y*' for '*x changes in y* ***accompanied by*** *muscular sensations*').

Second step

Further, he proceeds to a *conventional* classification of external changes: among external changes some can be compensated by an internal change, others cannot. If they can, experience teaches us only "that the compensation is approximately produced"; it gives the mind only "the occasion to accomplish this operation", but "the classification is not a raw fact of experience". If compensation is possible, the changes are called **changes of position**, if not, **changes of state**.

Third step

In this way, he obtains the following result: *modulo* an identity condition with respect to the compensation by internal changes, Poincaré defines the equivalence class of changes of position and calls it a *displacement*. Displacements form a group in the mathematical sense and it depends of the choice of its sub-groups whether the group corresponds to Euclidean or non Euclidean geometry.

Where does the concept of group come from?

At first glance, Poincaré's approach seems to be just an abstraction of the form of invariance. Nevertheless, in reality, the faculty to create the general concept of a transformation group is the expression of a form of our understanding "existing in our mind".

The set of relations satisfying the group axioms (the set theoretic model) is one expression of a structure, which is *exemplified* (in a Goodmanien sense) by the sensation system. In other words, the form in the mind is a special kind of an epistemologically accessible universal without that one have the possibility of deducing by purely logical means the particular form of the universal.

Exemplification + semantical density (Nelson Goodman)

Exemplification:

If a predicate *denotes* an object

(if homeomorphism denotes topological identity

the object can *exemplify* the predicate

red exemplifies (denotes iconically) the predicate « red »

(a cup and a tore exemplify homeomorphism)

The general transf. group structure as universal

The formation of the general structure of a transformation group — a 'universal' or a second order form (concept) —

$$G\ (A^1, A^2,)$$

is *suggested (exemplified)* by a specific sensation system

$$G'\ (S^1, S^2,)$$

which is the material of the form, i.e. a vague part of the extension of the concept which is for example a set theoretic model MG, i.e. a set equipped with the usual operations.

Intermediate Conclusion I

1) Poincaré begins his alternative reliability construction only *apparently* with sensations as ostensive contacts with the given. In reality, he introduces, similarly to Helmholtz's conception of intuition as imagined sensible impressions, a representation of a two-places sensation relation, based on the *imagination* of single sensations. Poincaré is not an empiricist.

2) The genesis of geometry is based on an epistemological process founded on previous classifications, carried out as a relationship between a structure as norm of invariance and a "conventional" adapted system as exemplification of these norms. The form of the group, i. e. the group structure, is suggested (exemplified) by the various imagined laws of sensations.

Intermediate Conclusion II

3) Read as relational set *MG* (i.e. extensionally), the general group is a model/interpretation of the group axioms and these axioms are instantiated by the *conventionally adapted* sensation system *G''*.

4) The elements of displacements groups are, as abstractions of sensations, complete and independent entities with respect to the axioms of the group, although empirically underdetermined.

Empirical *in*determination

According to Poincaré, the form of convention where exists a choice between different possibilities only becomes involved at a further step of the mathematical construction where the properties of the transformation group are studied:

We distinguish among the displacements belonging to groups isomorphic to G (among which some may operate on simpler material than the representative space) those that conserve certain sensations. The most intuitive are the subgroups of rotations. By taking these subgroups into considerations, we obtain a characterization of groups that correspond to geometries of constant curvature and decisions are token concerning the distance facing its empirical *indetermination* (the elements of the general group were only *under*determined).

From variables for complete objects to parameters

By analyzing the subgroups of the displacements groups, the variables of the axiom system are transformed into parameters that are depending on decisions that are token concerning the property of distance, for which exists a choice between different possibilities — such a choice was not yet considered with respect to the 'conventional' decisions with respect of the general group of transformation, *exemplified* by the imagined sensation relations: 'the combinations of external and internal changes form a group' (instantiation and iconic representation).

Poincaré's conception of structure I

Poincaré defends a structural point of view without completely disengaging the structure from an ostensional aspect what allows him to dispense with a consistency proof.

What matters here is the general idea *that the faculty of construction* of the general concept of group pre-exists in our minds and led to a universal, and that this faculty is suggested, i.e. exemplified, by a imagined system of sensations.

Poincaré's conception of structure II

The concept obtained can be read extensionally as a model of an axiom system, in which the domain of quantification is originally composed by independently given elements but which lose on the next step their independence and become incomplete.

The genesis of the geometrical metric *structure* is neither seen as the creation of the concrete material from the universal (structure) nor as the creation of the universal from the concrete (sensations), but as the advent of relations linked to the concrete in an *semiotic analysis* of the universal.

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Poincaré's conception of structure

- If my interpretation is right, then Poincaré's concept of structure is neither the new Hilbertian deriving from his axiomatization of geometry nor the traditional algebraic one.

Neither *in re* nor *ante rem*

Contrary to *in re* structuralists, Poincaré's universal, i.e. the general structure of a transformation group, is not ontologically but epistemically dependent of exemplifications. Moreover, Poincaré doesn't speak of this structure as such but *uses* it as a meta-mathematical *tool* for his psycho-physiological genesis of *real* actions with *imagined* sensations: they are the *ratio cognoscendi* of the existence of the faculty to build the general structure of a transformation group in our mind.

Structure and systems

The exemplification of the structure by large varieties of systems (in Poincaré: imagined sensations) being related by analogies is not a logical but a semiotic operation. The building up of the structure is a mastery occasioned by concrete systems (samples), which “show” or exemplify the structure. It’s only after a conventional decision that the elements of the exemplifying systems instantiate the axioms whose model has the structural properties. In this sense, the structure as tool gives the common character to systems.

The structure as a universal is not underdetermined but, as relational property, undetermined, i.e. essentially vague.

Pragmatic semantics (I)

On the basis of Peirce's pragmatism, we explain vagueness as indeterminacy of meaning in terms of indeterminacy of an exemplification of a concept in a dialogue (Williamson 1994].

In this sense the meaning of an 'object' or vague structure (second order relationship) of the form

$$R(P_1,\ldots,P_n) \approx \alpha'(P_1,\ldots,P_n) \wedge \beta'(P_1,\ldots,P_n) \wedge \ldots\ldots$$

(where '≈' means the exemplification of 'R', in a semiotic sense, by a system of axioms α', β' ...) has to be developed according the pragmatic maxim:

Peirce's pragmatic maxim

- "Consider what effects that might conceivably have practical bearing you conceive the object of your conception to have. Then your conception of those effects is the whole of your conception of the object" (C.P., 5.422; cf. Peirce 1878/79, p. 48 and C.P. 5.402).

The experimental perspective involved into the maxim is here performed by the exemplifications and limited by the constraints of definitions and of formalisms on the extensional level.

The semiotic stratification

The working out of universals as structures is still 'growing' and the extended structures still remain incomplete by principle, both, with respect to their proper identity and the identity of their objects distinguished by the structure.

This insight leads to an alternative interpretation of Quine's thesis of the incompleteness of mathematical objects and the ideas to which they belong: the incompleteness is neither an epistemic deficiency possessed finally by all objects according to Poincaré, nor a purely verbal accommodation, which in fact hides an ontological commitment with respect to a set theoretic progression (Quine 1986).

Resnik 1981, 546

We have in fact a *functional* peculiarity (relativity) of structures which gives an anachronistically confirmation of Resnik's result, quoted by Kate Hodesdon:

- [I have been taking] yet another approach to the question of what mathematical objects are […] My suggestion that mathematical objects are positions in patterns is not intended as a ontological reduction. […] My intention was instead to offer another way of viewing numbers and number theory which would put the phenomenon of multiple reductions and ontological and referential relativity in a clearer light.