Modal-Structural Mathematics in a Multiverse

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Foundations of Mathematical Structuralism

October 12-14, 2016
Munich

1 Review of Modal-Structural Interpretations

The aim of a modal-structural interpretation of a given mathematical theory is to capture the essential, structural content of the theory while "de-ontologizing" it, that is construing apparent commitment to special *mathematicalia* as elliptical for saying what *would* of necessity hold in any given structure of the appropriate type that there might be, while positing that indeed such a “structure” is mathematically possible. At the same time, even reference to *structures or to relations as objects* is *eliminated* in favor of reference to any suitably inter-related objects—regardless of their nature—that there might be according with the given axioms of the theory being interpreted. Importantly, the expressive resources of full, classical second-oder logic are available in the form of a combination of mereology and plural quantification (as articulted by Boolos in his "Nominalist Platonism", 1985).

Thus, the form of a modal existence postulate is as follows:

$$\Diamond \exists X, R_1^{n_1}, ..., R_k^{n_k} [Ax_T]^X_{P_1,...,P_k/R_1,...,R_k} \quad \text{(Cat-T)}$$

where X stands for a plurality of objects making up the universe of discourse, in superscript position indicating relativization of all quantifiers in the axioms Ax_T to that universe, and where the subscrpts indicate systematic replacement of the predicates P_i with the R_i abbreviating plural existential quantfication over n_i-tuples of items in X coding up n_i-ary relation $R_i^{n_i}$. (Burgess et al showed how to express ordered pairing using the machinery we assume here, viz. plural quantifiers + mereolgy. An infinity of pairwise non-overlapping objects is assumed in the background enabling this reduction. But that is an assumption that mathematics needs, in any case.) The formula displayed constitutes the categorical component of the MS-interpretation of theory T, assumed written with finitely many second- or first-order axioms, Ax_T.

The “hypothetical component”, i.e. the translation of sentences S of T, then takes the form of modal conditionals,

$$\Box\forall X, R_1^{n_1}, ..., R_k^{n_k}[Ax_T \rightarrow S]^X_{P_1,...,P_k/R_1,...,R_k}. \tag{Hyp-T}$$

Such translates nicely track proof-theoretic practice, articulating what successful proofs establish. In a sense, they reveal analytic status of the implications of the chosen axioms. Note however that, in view of Cat-T, this is *not* an "if-then" interpretation. Cat-T is an assertion of a mathematical possibility, implying the formal consistency of T and, even more, the possibility of a *standard* model for T, where that distinction is applicable. It cannot be claimed to be “analytic”. What about “a priori”? Even in the most basic instance of a natural-numbers structure, it can be doubted that we have incontrovertible *knowledge*—of the grade furnished by mathematical proofs—that a complete, “actual infinity” is possible. Rather, that is to be understood as a framework assumption, a working hypothesis that we take for granted in light of its simplicity and proven track-record. In this sense, there is room indeed for a "practical, empirical component" of purely mathematical existence postulates (in accordance with views of Quine and Putnam, but contrary to the logical empiricist tradition of Carnap, et al.)

2 Large Domains

It turns out that the combination of mereology and plural quantification can express what it is to be a domain D of set-like objects of strongly inaccessible cardinality. Illustration: operation of passing to the *power-set* of a set-like object. One requires that for any objects xx that are "few", i.e. not in 1-1 correspondence with all objects of D, there is a 1-1 correspondence between all the fusions of the xx and some atoms, aa. Clearly this can be iterated, taking all fusions of the aa, set in 1-1 correspondence with some further atoms, bb, and so on.

Similarly, the 2d-order Axom of Replacement can be expressed and guaranteed to hold of D. The combined effect of this and power-objects is that the cardinality of D is a regular, strong limit cardinal, i.e. strongly inaccessible.

Next, one can impose an *extendibility principle:* that any domain, satisfying axioms such as those of ZC, ZFC, etc. (in second-order form), has a proper extension to a larger domain, in abbreviated form:

$$\Box\forall\mathcal{M}\Diamond\exists\mathcal{M}'[\mathcal{M} \prec \mathcal{M}'] \tag{EP}$$

(This goes back to Zermelo [1930], and was given a modal expression independently by Putnam [1967].) It is straightforward to iterate these moves to obtain hyper-inaccessibles, Mahlo cardinals, and their ilk.

Note that the expressive resources are available to introduce small categories and toposes as well. Indeed, our second-order machinery recovers Grothendieck's method of universes. Moral here: set theory and category theory can be developed side-by-side, without one being taken as more fundamental than the other.

3 2d-order Reflection Scheme

This has the following form:

$$\forall X[\phi(X) \to \exists\beta(\phi^{\beta} X \cap R(\beta))], \qquad \text{(Refl 2d-order)}$$

where $\phi^{\beta}(X)$ is the result of restricting the first- and second-order bound variables in $\phi(X)$ to the ranks $R(\beta)$ and $R(\beta+1)$ respectively. What is the motivation for such a principle? And what are its consequences?

The usual motivation, in abstract, actualist terms, is that the fixed background universe of absolutely all sets and ordinals, V, is so vast as to be *indescribable,* meaning that any condition holding over that whole universe also already holds of an initial segment of that universe, i.e. the condition holds with all first- (second-) order quantfiers relativized to rank $R(\beta)$ ($R(\beta+1)$).

Regarding consequences, adding the 2d-order reflection scheme to Zermelo set theory without Infinity results in a vast simplification through unification: for this theory implies Infinity, 2d-order Replacement, Inaccessible and Mahlo cardinals of all orders, and even the indescribable cardinals, indeed, stationery sets thereof. That's a remarkable unification that any foundational program should seek to retain.

How can the MS program do this, however, since it forswears any reference to a fixed, maximal background universe, promoting instead the extendability principle without restriction?

Well, one may ask, how does MS pass from (the possibility of satisfying) Zermelo set theory to (the possibility of satisfying) Zermelo-Fraenkel, with the 2d-order Replacement Axiom? Motivated by primarily mathematical considerations, e.g. to obtain a reasonable theory of ordinal arithmetic, one adds that we're interested in investigating structures that extend upward beyond what can be measured by anything occurring at a rank, i.e. the structure of ranks reaches higher than can be measured by any single rank. Thus we articulate a largeness condition that describes the sort of structures we're interested in investigating; and, in the case of Replacement, we can note that that condition already holds of the structure of hereditarily finite sets, bolstering our confidencwe in its consistency. To be sure, this is short of the kind of knowledge conferred by mathematical proof, but that is in the nature of new axioms.

I suggest here that a similar story can be told motivating the mathematical possibility of 2'd-order Reflection:

(1) We're interested in studying structures so vast as to be indescribable, in the sense of satisfying 2'd-order Reflection;

(2) The implications result in a vast simplificiation and unification of weaker principles motivating small large cardinals.

Neither of these points even suggests, much less implies, that a fixed, maximal, inextendable background universe is even a possibility.

Thus we adopt directly:

$$\Diamond\exists\mathcal{M}\forall X\subseteq\mathcal{M}[\phi(X)\rightarrow\exists\beta(\phi^{\beta}X\cap R(\beta))],$$

(Poss 2d-order Refl.)

where all quantifiers are relativized to the posited domain of $\mathcal{M}$, taken to be a model of Zermelo set theory less Infinity.

Note that this method of direct postulation, bypasses any appeal to “ultimate infinity”, either in the form of a fixed, maximal universe of discourse, or even of a range of “all possible” extensions of a given structure for set theory. On the contrary, we take the idea of "ultimate infinity" to be mathematically incoherent, indeed belied by the unrestricted extendability principle (expressed in modal form, as Putnam (1967) first suggested).

Finally, it is instructive to note that a natural modal reflection principle appealing to Putnam's extendability interpretation of unbounded set-theoretic sentences, while it demonstrably works faithfully for first-order sentences, breaks down for second-order. (This has been demonstrated by Sam Roberts.) For example, consider the axiom of inaccessibles, viz., abbreviated by

$$\forall\alpha\exists\beta[\beta > \alpha \wedge Inac(\beta)].$$

The Putnam extendability translation of this takes the form,

$$\Box\forall M, \alpha \text{ in } M \Diamond\exists M', \beta \text{ in } M'[M' \succ M \wedge \beta > \alpha \wedge Inac(\beta)],$$

where α, β range over ordinals and M, M' range over standard models of ZF^2C. Now note that the latter follows from the EP along with Zermelo's theorem on the inaccessible height of such standard models. A strong modal reflection principle would then take us from this latter formula to the conclusion that, at some standard model of ZF^2C, the original axiom holds. Clearly, this pattern can be iterated to obtain higher-order inaccessibles and much morel

That's the good news. The bad news, however, is that *too much* more follows, viz. outright inconsistency, once the translation scheme is extended to second-order sentences (based on a clever application of Gödel incompleteness, due to Sam Roberts). Now it may be possible to restrict the modal reflection principle so as to weed out such counterexamples, but it is not evident how to do so. But, as already suggested, even though the scheme would indeed deliver the possibility of a standard model of 2d-order Reflection, it is better just ot posit the latter outright, based on the mathematical interest in investigating large universes, rather than appealing to an ultimate infinity of (in the present setting) possibilities of models. The Putnam translation scheme is useful and insightful in the case of first-order ZFC; but it appears to go too far when one attempts to apply it to second-order sentences.