A Theory of Unlabeled Graphs as *Ante Rem* Structures

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Mathematics as a science of structures:

> *If in the consideration of a simply infinite system N set in order by a transformation φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation, then are these elements called natural numbers... (Dedekind 1888)*

Dedekind on arithmetic and analysis
Hilbert on geometry
Zermelo on set theory

(Quasi-)Categoricity, if second-order

Algebra
Logic
Bourbaki
Category theory
Homotopy type theory (univalence)

General structures (incl. "algebraic")

Structuralism in the philosophy of mathematics answers ontological, semantic, and epistemological questions about mathematics in a structuralist manner.

> *[I]n giving the properties. . . of numbers you merely characterize an abstract structure. . . [T]he "elements" of the structure have no properties other than those relating them to other "elements" of the same structure. . . To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, to be followed by 4, 5, and so forth. . . (Benacerraf 1965)*

Goal:

- Make one version of structuralism—*ante rem* structuralism—more precise in terms of an example theory and argue that it is coherent.

Plan:

1. Variants of Mathematical Structuralism
   (*not* complete: e.g., Awodey 1996)
2. A Theory of Unlabeled Graphs as *Ante Rem* Structures
3. Philosophical Assessment of the Theory
4. Conclusions

We will concentrate on ontological and semantic questions about structures: *what are structures*, and *how can we speak about them*?

Variants of Mathematical Structuralism (Reck & Price 2000)

Eliminativist: structure talk concerns instantiating representations/systems.

- Relativist:

  $0 = \varnothing$
  $1 = \{\varnothing\}$
  $2 = \{\varnothing, \{\varnothing\}\}$
  $\vdots$

  $\mathbb{N} = \omega$
  $succ(n) = n \cup \{n\}$

  E.g., '$succ(0) \neq 0$': $\varnothing \cup \{\varnothing\} \neq \varnothing$

  $\langle \mathbb{N}, 0, succ \rangle$ satisfies Dedekind-Peano's categorical $PA_2(N, \overline{0}, s)$.
  The choice of set-theoretic representation is *arbitrary up to isomorphism*.

  What an arithmetical statement says is relative to representation.

- Universalist:

  E.g., '$succ(0) \neq 0$':

  $$\forall X, y, f(PA_2(X, y, f) \rightarrow f(y) \neq y)\ [\wedge\ \exists X, y, f\, PA_2(X, y, f)].$$

  An arithmetical statement is really a quantified statement about *all set-theoretic representations* that satisfy the Dedekind-Peano axioms.

  (There are also nominalist & modal versions: see Hellman 1989.)

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  E.g., '$succ(0) \neq 0$':

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	Relativist	Universalist
Ontological	too much structure (e.g., $0 \in 1$)	✓
Semantic	✓	$N, \overline{0}, s$ not at face value

Non-eliminativist and *ante rem*: structures are "*independent*" of instantiations.

Structures are abstract forms that their instantiations have in common.

But structures also consist of individuals, relations, and functions.
However, these components "have no identity or distinguishing features outside a structure" (Resnik 1997; see also Shapiro 1997, Parsons 1990).

E.g., there is the unique *ante rem* structure of natural numbers:

'$succ(0) \neq 0$': $succ(0) \neq 0$

	Ante rem
Ontological	✓ (right "amount" of structure)
Semantic	✓ ($N, \overline{0}, s$ taken at face value)

But what makes the arithmetical structure a *structure*? What are structures?

How can ‘0’ refer to 0, ‘*succ*’ refer to *succ*, etc.?

Is such a conception of *ante rem* structures even coherent?

> *It is impossible that the [numbers] should be, as Dedekind suggests, nothing but the terms of such relations as constitute a progression. If they are to be anything at all, they must be intrinsically something (Russell 1903)*

(And there are also various contemporary philosophers who question the coherence of *ante rem* structuralism.)

A Theory of Unlabeled Graphs as *Ante Rem* Structures

Unlabeled graph: a graph in which individual nodes have no distinct identifications except through their interconnectivity. (Wolfram MathWorld)

two graphs $G = (N, E)$ and $H = (N, F)$ are the same unlabeled graph when they are isomorphic... (Mahadev & Peled 1995, Threshold Graphs and Related Topics)

Sometimes we are interested only in the “structure” or “form” of a graph and not in the names (labels) of the vertices and edges. In this case we are interested in what is called an unlabeled graph. A picture of an unlabeled graph can be obtained from a picture of a graph by erasing all of the names on the vertices and edges. This concept is simple enough, but is difficult to use mathematically because the idea of a picture is not very precise. (Bender & Williamson 2010, Lists, Decisions and Graphs)

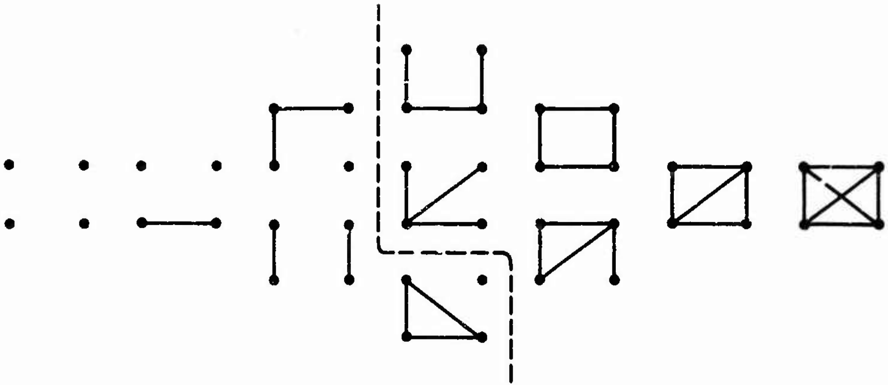


Fig. 2.1. The graphs with four points.

(Harary 1969, *Graph Theory*)

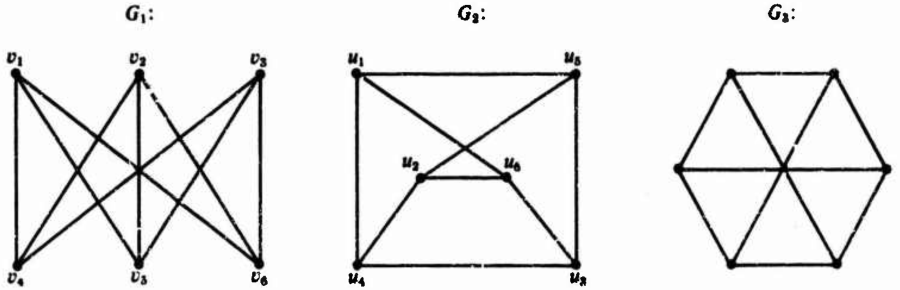


Fig. 2.5. Labeled and unlabeled graphs.

A graph G is labeled when the p points are distinguished from another by names such as v_1, v_2,..., v_p, *For example, the two graphs* G_1 *and* G_2 *of Fig. 2.5 are labeled but* G_3 *is not.*

Rather than continue with an intuitive development of additional concepts, we proceed with the tedious but essential sequence of definitions upon definition. (Harary 1969)

We seem to have a clear intuition (*Anschauung*) of unlabeled graphs.

But that intuition is not preserved by the set-theoretic definition of 'graph':

- A graph is a pair $\langle V, E\rangle$, such that $V \neq \emptyset,\; E \subseteq \{\{v,w\} | v,w \in V, v \neq w\}$.

  E.g.:

  $\langle\{0,1,2\},\{\{0,1\},\{1,2\}\}\rangle$ (= $\{\{\{0,1,2\}\},\{\{0,1,2\},\{\{0,1\},\{1,2\}\}\}\}$).

  $\langle\{\pi,e,0\},\{\{\pi,e\},\{e,0\}\}\rangle$ (= $\{\{\{\pi,e,0\}\},\{\{\pi,e,0\},\{\{\pi,e\},\{e,0\}\}\}\}$).

- A labeled graph is a triple $\langle V, E, l\rangle$, such that $\langle V, E\rangle$ is a graph and $l\colon V \to \mathbb{N}$.

In the following, we are going to state an axiomatic theory UGT of unlabeled graphs (undirected, without loops or multiple edges) as structures *sui generis*.

Language:

- language of second-order logic with identity.

  Primitive predicates: '$Graph(G)$', '$Vertex(v, G)$', '$Connected(v, w, G)$'.

  'G', 'v': first-order variables (sometimes we use 'G' restricted to graphs).

  'X', 'R', 'f': second-order variables.

Intended first-order universe D:

- unlabeled graphs and their vertices.

Intended second-order universe:

- sets, relations, functions on D.

Logic: standard deductive system of second-order logic; in particular:

- $\exists X \forall x (X(x) \leftrightarrow \varphi[x])$ (X not free in φ).

  E.g.: $\forall G \exists X \forall x (X(x) \leftrightarrow \mathit{Vertex}(x, G))$.

  φ is functional $\rightarrow \exists f \forall v, w (f(v) = w \leftrightarrow \varphi[v, w])$ (f not free in φ)

- $\forall x, y$: $x = y \leftrightarrow \forall X (X(x) \leftrightarrow X(y))$.
  $\forall X, Y$: $X = Y \leftrightarrow \forall x (X(x) \leftrightarrow Y(x))$.

  E.g.: $\forall G \exists ! X \forall x (X(x) \leftrightarrow \mathit{Vertex}(x, G))$.

  $\forall f, g$: $f = g \leftrightarrow \forall x (f(x) = g(x))$.

- Choice Axiom:
  $\forall R^{n+1} (\forall x_1, \ldots, x_n \exists y R^{n+1}(x_1, \ldots, x_n, y) \rightarrow$
  $\exists f^n \forall x_1, \ldots, x_n R^{n+1}(x_1, \ldots, x_n, f(x_1, \ldots, x_n)))$.

Various definitions, e.g.:

- $\forall v, \forall G$: $\mathit{Isolated}(v, G) \leftrightarrow \mathit{Vertex}(v, G) \wedge \neg \exists w\, \mathit{Connected}(v, w, G)$.
- $\forall G, \forall X$: $V(G) = X \leftrightarrow \forall x (X(x) \leftrightarrow \mathit{Vertex}(x, G))$.

General axioms for unlabeled graphs:

- $\forall G \forall v, w\colon \mathit{Connected}(v, w, G) \rightarrow$

  (i) $\mathit{Vertex}(v, G) \wedge \mathit{Vertex}(w, G)$,

  (ii) $v \neq w$,

  (iii) $\mathit{Connected}(w, v, G)$.

- $\forall G \forall v\colon \mathit{Vertex}(v, G) \rightarrow$

  $\neg \exists G' (G' \neq G \wedge \mathit{Vertex}(v, G')) \wedge \neg \mathit{Graph}(v)$.

- Identity criterion: $\forall G, G'\colon G = G' \leftrightarrow G \cong G'$.

  With:

  - $G \cong G' \leftrightarrow \exists f (f\colon G \rightarrow G' \wedge f \text{ bijective}_{G,G'} \wedge f \text{ structure-preserving}_{G,G'})$.
  - $f\colon G \rightarrow G' \leftrightarrow \forall v (\mathit{Vertex}(v, G) \rightarrow \mathit{Vertex}(f(v), G'))$.
  - $f \text{ bijective}_{G,G'} \leftrightarrow \forall w (\mathit{Vertex}(w, G') \rightarrow \exists! v (\mathit{Vertex}(v, G) \wedge f(v) = w))$.
  - $f \text{ structure-preserving}_{G,G'} \leftrightarrow \forall v, w (\mathit{Vertex}(v, G) \wedge \mathit{Vertex}(w, G) \rightarrow$ $(\mathit{Connected}(v, w, G) \leftrightarrow \mathit{Connected}(f(v), f(w), G')))$.

Existence axioms for unlabeled graphs:

- $\exists G \exists! v\, Vertex(v, G)$.

  The existence of the trivial graph (Harary, p.9).

- $\forall G \exists G' \exists v'$, such that:
  $Vertex(v', G')$,
  $Isolated(v', G')$,
  $\exists f\; Isomorphism(f, G, G' - v')$.

  $G' - v'$: removal of a point (Harary, p.11).

- $\forall G \forall v, w\, (Vertex(v, G) \wedge Vertex(w, G) \wedge v \neq w \wedge \neg Connected(v, w, G) \rightarrow$
  $\exists G' \exists v' \exists w'$, such that:
  $Connected(v', w', G')$,
  $\exists f(Isomorphism(f, G, G' - \{v', w'\}) \wedge v' = f(v) \wedge w' = f(w))$.

  $G' - \{v', w'\}$: removal of a line (Harary, p.11).

On this basis we can prove all finite unlabeled graphs to exist, and we can determine the cardinality of unlabeled graphs with a fixed number of vertices.

Theorem

E.g., using UGT, we can derive:

- $\exists! G_0 \exists! v \, Vertex(v, G_0)$.
- $\exists G_1 \exists v_1, v_2 (v_1 \neq v_2 \land Vertex(v_1, G_1) \land Vertex(v_2, G_1) \land \neg Connected(v_1, v_2, G_1) \land \forall w (Vertex(w, G_1) \rightarrow w = v_1 \lor w = v_2))$.
- *There exist precisely four unlabeled graphs with three vertices.*

Proof: Combination of existence axioms and identity criterion.

(All of these finite unlabeled graphs can be described categorically, of course.)

Accordingly, one can determine the right number of automorphisms for a given unlabeled graph; and so on.

E.g.: $\exists G_1 \exists v_1, v_2 (v_1 \neq v_2 \wedge Vertex(v_1, G_1) \wedge Vertex(v_2, G_1) \wedge \neg Connected(v_1, v_2, G_1) \wedge \forall w (Vertex(w, G_1) \rightarrow w = v_1 \vee w = v_2))$.

Proof (sketch of main steps):

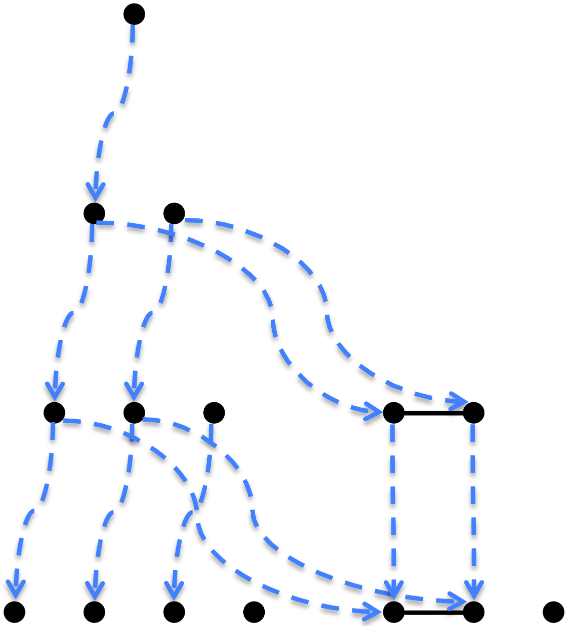
Ax. $\exists G \exists ! v Vertex(v, G)$.

EI $Vertex(v_0, G_0) \wedge \forall w (Vertex(w, G_0) \rightarrow w = v_0)$.

Ax. $\forall G \exists G' \exists v'$, such that:
$Vertex(v', G') \wedge$
$\exists f (\forall v (Vertex(v, G) \rightarrow Vertex(f(v), G') \wedge f(v) \neq v') \wedge$
$\forall w' (Vertex(w', G') \wedge w' \neq v' \rightarrow \exists ! w (Vertex(w, G) \wedge f(w) = w')) \wedge$
$\forall v, w (Vertex(v, G) \wedge Vertex(w, G) \rightarrow (Connected(v, w, G) \leftrightarrow$
$Connected(e', f(v), f(w), G')))) \wedge$
$Isolated(v', G')$.

UI/EI $Vertex(v_1, G_1) \wedge Vertex(f(v_0), G_1) \wedge f(v_0) \neq v_1 \wedge$
$\forall w' (Vertex(w', G_1) \wedge w' \neq v_1 \rightarrow f(v_0) = w') \wedge$
$Isolated(v_1, G_1)$.

EG Done.



Metatheorem

UGT is consistent.

Proof: Provide set-theoretic model.

- First-order domain D: For each isomorphism type of finite set-theoretic graphs with vertices in $\mathbb{N}$, pick one member; but do so in a way such that no two picked set-theoretic graphs share a vertex.

  Put these set-theoretic graphs into D as well as their vertices.

- Second-order domain: all sets, relations, functions on D.

- Interpret '$Graph(G)$', '$Vertex(v, G)$', '$Connected(v, w, G)$' as expected.

Extensions of the system:

- Introduce natural numbers and functions from vertices to natural numbers: state the second-order Dedekind-Peano axioms; include the natural numbers in the intended universe.

  Based on this, we can define, e.g.:

  f is a walk in G iff $\exists x(Nat(x) \wedge \forall y(Nat(y) \wedge y \leq x \rightarrow Vertex(f(y), G)) \wedge$
  $\forall y(\neg Nat(y) \vee y > x \rightarrow y = G_0) \wedge$
  $\forall y(Nat(y) \wedge y < x \rightarrow Connected(f(y), f(y+1), G)))$.

  Define: connectedness, length of walk, distance, degree, etc.

  One can define recursive functions on graphs explicitly, prove theorems by induction (e.g. over the number of vertices of graphs), and derive in this way theorems about all finite unlabeled graphs.

Relations between graphs:

- G' is a subgraph of G if and only if $\exists X \exists f(\forall v(X(v) \rightarrow Vertex(v, G)) \wedge Isomorphism(f, G|_X, G'))$.

Further graph-theoretic operations:

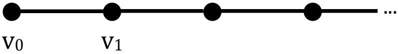
- Subgraph axiom:

  $$\forall G \forall X\,(\forall v(X(v) \rightarrow Vertex(v, G)) \rightarrow \exists G' \exists f\, Isomorphism(f, G|_X, G')).$$

  $Isomorphism(f, G|_X, G')$:

  $$\begin{aligned}
  &\forall x(X(x) \rightarrow Vertex(f(x), G')) \wedge \\
  &\forall x'(Vertex(x', G') \rightarrow \exists! x(X(x) \wedge f(x) = x')) \wedge \\
  &\forall x, y(X(x) \wedge X(y) \rightarrow (Connected(x, y, G) \leftrightarrow \\
  &\qquad\qquad\qquad\qquad Connected(f(x), f(y), G'))).
  \end{aligned}$$

- Union graph axiom (not literal union!)
- Product graph axiom (use category-theoretic formulation!)
- $\vdots$



- Infinity graph axiom:

  $\exists G \exists v_0, v_1\colon Vertex(v_0, G) \land Vertex(v_1, G) \land Connected(v_0, v_1, G) \land$
  $\forall w(Connected(w, v_0, G) \rightarrow w = v_1) \land$
  $\exists f(Isomorphism(f, G, G - v_0) \land f(v_0) = v_1)$.

($\rightarrow$ There exists a special graph which is *identical* to one of its subgraphs.)

Use subgraph axiom to determine the (structurally) least graph of that kind.

Philosophical Assessment of the Theory

This theory of unlabeled graphs should count as an axiomatic treatment of (a special family of) *ante rem* structures:

> *If in the consideration of a simply infinite system N set in order by a transformation φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation, then are these elements called natural numbers… (Dedekind 1888)*

Now we discuss some of the standard worries about *ante rem* structuralism against the background of our theory.

Ontological and semantic worries:

- Resnik (1997) takes identities across structures to be indeterminate, while Shapiro (2006) takes them to be false (as do Linsky & Zalta 2006).

  But this may run counter to mathematical practice: cf. MacBride (2005).

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  – Not much of an issue for unlabeled graphs:

  Graph theorists do not identify vertices from distinct unlabeled graphs. ✓

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  – Even if they did in some cases, this would not be a problem:

  Drop: $\forall G \forall v(Vertex(v, G) \rightarrow \neg\exists G'(G' \neq G \wedge Vertex(v, G')))$.

  Deductively, it was not used anyway.

  Since connectedness is graph-relative ('$Connected(v, w, G)$'),
  and since all constructions had been carried out "along isomorphisms",
  allowing for "crossroad vertices" would be consistent.

- If two objects a, b in a structure are structurally indistinguishable—that is, there is an automorphism f so that $f(a) = b$—shouldn't they be identical?

  Which would be against mathematical practice again (e.g., i vs $-i$).
  cf. Burgess (1999), Keränen (2001).

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  $\hookrightarrow$ *No!* (cf. Ketland 2006, Leitgeb & Ladyman 2008, Shapiro 1997, 2008.)

Let a, b so that $Vertex(a, G_1)$, $Vertex(b, G_1)$, and $a \neq b$:

the fact that $a \neq b$ obtains in virtue of

$$\exists v_1, v_2(v_1 \neq v_2 \wedge Vertex(v_1, G_1) \wedge Vertex(v_2, G_1) \wedge \neg Connected(v_1, v_2, G_1) \wedge \forall w(Vertex(w, G_1) \rightarrow w = v_1 \vee w = v_2)).$$

That $a \neq b$ holds is grounded in what the unlabeled graph G_1 is like. ✓

Accordingly, our logical identity principles are perfectly consistent with our *ante rem* structuralism about unlabeled graphs:

– $\forall x, y$: $x = y \leftrightarrow \forall X(X(x) \leftrightarrow X(y))$.

Don't restrict PII to "qualitative" properties, don't think of it predicatively.

Even Shapiro (2008) gets it wrong on identity properties: identity is perfectly structural, just as *number of vertices in graph G* is!

– $\forall X, Y$: $X = Y \leftrightarrow \forall x(X(x) \leftrightarrow Y(x))$.

Don't misunderstand this to be about identity of *subgraphs*!
Within a G, 'X' and 'Y' range over *sets* of vertices or, plurally, over *vertices*.

– $\forall f, g$: $f = g \leftrightarrow \forall x(f(x) = g(x))$

Which yields, e.g., the right number of automorphisms for graphs. ✓

- “Objects” (“places”) in *ante rem* structures are not really *objects*.

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- "Objects" ("places") in *ante rem* structures are not really *objects*.

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  $\hookrightarrow$ Vertices in unlabeled graphs are not substances/individuals in a traditional metaphysical sense (cf. Caulton and Butterfield 2012), but they are objects in a logical or Quinean sense:

  – they are (members of) values of bound variables ($\exists v$, $\forall v$, $\exists X$, $\forall X$),

  – one can map them to other objects,

  – there is an identity/difference relation for them,

  – one can count them. ✓

- In order to clarify non-eliminative structuralism, we need a notion of *"important" property* that is preserved when we abstract out structures from instantiating systems.

  Compare Øystein's talk from yesterday.

  $\hookrightarrow$ We did not need any such notion since we did not introduce unlabeled graphs via abstraction. Unlabeled graphs are *primitives*. $\checkmark$

- There are no precise axioms for *ante rem* structures, or such axioms are just “set theory in disguise” (such as the axioms in Shapiro 1997).

  cf. Nodelman & Zalta (2014), Hellman (2005).

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  $\hookrightarrow$ Done. ✓

The axiomatic system UGT for unlabeled graphs is

(i) in line with pre-set-theoretic mathematical practice,
(ii) exact,
(iii) based on a structuralist identity criterion,
(iv) consistent,
(v) can easily be strengthened.

(What more do you want me to do?)

- Is it possible to refer to all of the individuals in non-rigid structures (where there exists an automorphism f, such that $f(a) = b$ and $a \neq b$)?

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Informally: Let 'a' denote any node in G_1; let 'b' denote the other node.

Formally: $a = \varepsilon v Vertex(v, G_1)$; $b = \varepsilon v(Vertex(v, G_1) \wedge v \neq a)$.

cf. Hilbert, Bourbaki, Shapiro (2008) on epsilon terms.
Or think of epsilon terms as used only in the act of baptism (Kripke!).

Given the ε-term definitions of 'a' and 'b', our previous 'the fact that $a \neq b$ obtains in virtue of...' can be made precise in terms of 'is derivable from'.

Even Shapiro (2008) gets this wrong when he says "There simply is no naming any point... in some graphs...":

There are simply two distinct but structurally indistinguishable reference relations on G_1 above, just as it had been the case for *a* and *b* themselves.

(Or maybe this is not "real" reference? Well,...)

Conclusions

- Unlabeled graphs can be treated, mathematically and philosophically, as *structures sui generis*.

  (Interestingly, when one does so, the boundaries between *doing* mathematics structurally and structuralism *about* mathematics become fuzzy.)

- At least as far as unlabeled graphs are concerned, *ante rem* structuralism amounts to a coherent position in the philosophy of mathematics.

- None of this is against set theory *per se*: just against taking the set-theoretic reconstruction of the *basic structures* too seriously.

Unlabeled Graph Theory vs Set Theory

How does UGT relate to set theory?

Set theory has three roles to play in modern mathematics:

1. Set theory as a mathematical "language":

   e.g., second-order quantifiers $\forall X$, $\exists X, \forall f, \exists f$ in UGT. ✓

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   the mathematical study of the cumulative hierarchy and related structures (mostly independent of UGT). $\checkmark$

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2. Set theory as a special area of mathematics:

   the mathematical study of the cumulative hierarchy and related structures (mostly independent of UGT). $\checkmark$

3. Set theory as a foundation of mathematics:

   Interpreted axiomatic set theory plus methods of reducing mathematical objects, concepts, theorems to set-theoretic objects, concepts, theorems.

   How this squares with UGT depends on the *purpose* of reduction.

Set theory as a foundation of mathematics:

3a For *mathematical purposes:* show that reduction is possible and how so.

Relative interpretability yields (i) relative consistency, (ii) a way of using methods and results from one field in another, and more.

E.g., proof of consistency of UGT ($\approx$ relativist structuralism!). ✓

With a bit more work, the direction of reduction might also be reversed (standard set theory as theory of *pointed, grounded, directed graphs under bisimilarity*; cf. Aczel 1988).

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3b For *quasi-philosophical purposes:* determine purely set-theoretically (what should be) the intended interpretation of the language of mathematics.

This I find questionable! $\not\checkmark$

Relative interpretation preserves derivability but not necessarily meaning.

is *simulated* by $\{\{\{0,1,2\}\},\{\{0,1,2\},\{\{0,1\},\{1,2\}\}\}\}$
(or its isomorphism class)