Pure structure as ‘one over many’

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Introduction

Structuralism $:=$ non-eliminative, mathematical structuralism

The structuralist slogan

Mathematical objects are *just* positions in structures.

Systems and pure structures

A *system* is 'a collection of objects with certain relations' (Shapiro, 1997, p. 73): a domain D and relations $R_1, \ldots, R_n$ on this domain.

A *pure structure* is

> *the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system (Shapiro, 1997, p. 74).*

A pure structure is also said to be a 'one over many'.

My plan

- Three completely different explications of the idea of a pure structure as a 'one over many'.
- Nevertheless, there are deep connections between the explications.
- This points the way to a better understanding of pure structures.

Dedekind on 'foreign' properties

I would advise that by number one understand not the class itself (the system of all finite systems that are similar to each other) but something new [...] which the mind creates. [...] [O]ne will say many things about the class (e.g. that it is a system of infinitely many elements, namely, of all similar systems) that one would apply to the number only with the greatest reluctance; does anybody think, or won't he gladly forget, that the number four is a system of infinitely many elements? (But that the number four is the child of the number three and the mother of the number five is something that nobody will forget.)
(Dedekind, letter to Weber, 24.01.1888)

Dedekind abstraction

Dedekind called these ‘foreign properties’ and held that an account that requires us to ascribe such properties to the relevant objects is inadequate (Dedekind, 1963, 10).

As Frege joked in a related context, we need a lye that is just strong enough to wash away all of the unwanted properties, while preserving all of the desired ones. (Frege, 1894, p. 84)

Dedekind abstraction: the desiderata

For every system S, there is a pure structure $[S]$ such that

- *Instantiation.* S is isomorphic to $[S]$.
- *Purity.* Every property of an element of $[S]$ is structural, not foreign.
- *Uniqueness.* $[S]$ is unique in satisfying *Instantiation* and *Purity*.

Counterexamples to Purity

(i) intentional properties: e.g. being Gauss's favorite number

(ii) applied properties: e.g. being the number of bicycles that I own

(iii) metaphysical properties: e.g. being abstract

(iv) kind properties: e.g. being a natural number

Restricted Purity

Every "important" property of an element of a pure structure is structural.

Now we need a definition of "important" that makes this claim both true and non-trivial.

Pure structure as types and as universals

> *Thus, structure is to structured as pattern is to patterned, as universal is to subsumed particular, as type is to token. (Shapiro, 1997, 84)*

Three ways in which a pure structure can be a 'one over many'

- The result of *Dedekind abstraction*, i.e. washing away the foreign properties.
- A *type*, contrasted with the systems as tokens. Cf. Fregean abstraction:

$$\mathrm{type}(x) = \mathrm{type}(y) \leftrightarrow x \sim y$$

- A *universal*, i.e. something that is predicated of each of the systems.

A muddle?

Dedekind abstraction

- What is Dedekind's 'metaphysical lye'?
- Counterexamples to Purity (or must define "important")

Types

> *Sometimes [pure structures] are confused with isomorphism types, but this is a mistake: An isomorphism type is no more a special kind of system than a direction is a special kind of line. (Burgess, 1999, pp. 286–7)*

Universals

- If a pure structure is a universal and also satisfies Instantiation, the universal would have to instantiate itself. But universals don't ordinary self-instantiate.

Three individually problematic ideas conflated in an awful muddle?

A warm-up case: Frege on directions

$$d(l_1) = d(l_2) \leftrightarrow l_1 \parallel l_2 \tag{Dir}$$

Some properties of lines are **inherited** by their directions, e.g.

$$d(l_1) \perp^* d(l_2) \leftrightarrow l_1 \perp l_2$$

More generally, for any relation R on lines that "respects parallelism" there is an associated relation R^* on directions.

Other properties of lines are **not inherited**, color, location, etc.

A warm-up case: the unification

Unifying Frege abstraction and Dedekind-style 'purification'

- Directions have inherited and non-inherited properties.
- They have been purified with respect to the former.

Unifying Frege abstraction and universals

- Consider the universal $U_l(x)$ of being parallel to l. The universals of this form partition the domain of lines into equivalence classes.
- For each U_l there is an abstract $d(l)$ whose inherited properties are precisely those that are shared by each instance of the universal U_l:

$$\forall x(U_l(x) \to Fx) \quad \text{iff} \quad F^*(d(l))$$

- Thus, $d(l)$ can be seen as a "generic instance" of the universal U_l whose inherited properties are all and only those shared by all instances of the universal.

Dedekind abstraction: preliminaries

Consider a system $S = \langle D, R_1, \ldots, R_n \rangle$. We say this is an *$\mathcal{L}$-system* when it can be regarded as an $\mathcal{L}$-model, i.e. all the arities match.

We would like to postulate a corresponding pure structure $[S] = \langle [D]_S, [R_1]_S, \ldots, [R_n]_S \rangle$, where $[D]_\mathcal{S} = \{[a]_\mathcal{S} : a \in D\}$.

We'd like to satisfy the three desiderata, in particular

$$[S] = [S'] \leftrightarrow S \cong S'$$

Isomorphism invariance: In all of our definitions, we must ensure it doesn't matter whether we approach $[S]$ via S or via some isomorphic S'.

An illustration of what we want

Example

Let $\mathcal{L}$ be the first-order language with $<$ as its sole non-logical predicate.

- *$\mathcal{S}$ consists of three balls ordered by strictly increasing mass, namely a, b, c, in that order.*
- *$\mathcal{S}'$ consists of the same three balls ordered by strictly increasing volume, namely b, a, c, in that order.*

The resulting pure structure $[\mathcal{S}] = [\mathcal{S}']$ is based on three "pure positions": being first, second, and third, which are realized the balls.

There is a "pure relation" $<$ on this domain, which is realized by the two impure relations in $\mathcal{S}$ and $\mathcal{S}'$.

Pure positions and their definable properties

We *postulate* a pure position $[a]_S$ for each a in the domain D of S. So far, these items are "made of" just syntax and material from $\mathcal{S}$!

Suppose $\phi \in \mathcal{L}$. We define what it is for some pure positions **derived from one and the same system** $\mathcal{S}$ to have a **definable property**:

$$\phi([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) \quad \text{:iff} \quad \mathcal{S} \models \phi(a, b) \tag{D1}$$

Notice that (D1) yields $[a]_{\mathcal{S}} = [b]_{\mathcal{S}}$ iff $\mathcal{S} \models a = b$. So the pure positions from $\mathcal{S}$ are individuated **relative to one another**.

Definition (D1) is isomorphism invariant

Proposition (Isomorphism Theorem)

Let $\mathcal{S}$ and $\mathcal{S}'$ be $\mathcal{L}$-systems, and let $\phi(\vec{x})$ be an $\mathcal{L}$-formula. Assume $f : \mathcal{S} \to \mathcal{S}'$ is an isomorphism. Then:

$$\mathcal{S} \models \phi[\vec{a}] \quad \textit{iff} \quad \mathcal{S}' \models \phi[f(\vec{a})]$$

This means that definition (D1) is invariant.

Recall what the definition says:

$$\phi([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) \quad :\text{iff} \quad \mathcal{S} \models \phi(a, b) \tag{D1}$$

The Instantiation requirement is satisfied

We define a mapping $S \to [S]$ by letting $a \mapsto [a]_S$. That is, we map an object to the pure position that this object occupies in $\mathcal{S}$.

Definition (D1) ensures

$$P_i([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) \quad \text{iff} \quad R_i(a, b)$$

Thus, $\mathcal{S} \cong [\mathcal{S}]$. (Recall R_i is the interpretation in $\mathcal{S}$ of the predicate P_i.)

For instance, in our ball example we have:

$$[a]_{\mathcal{S}} \leq [b]_{\mathcal{S}} \quad \text{iff} \quad a \leq_{\text{mass}} b$$
$$[b]_{\mathcal{S}'} \leq [a]_{\mathcal{S}'} \quad \text{iff} \quad b \leq_{\text{volume}} a$$

Taking stock

We have defined a logically coherent way to talk about pure positions **derived from one and the same system** and their **definable** properties. What is the *metaphysical significance* of our definition?

Eliminativism: just a *façon de parler*.

Very useful: handles all ordinary mathematical talk of pure structures.

Non-eliminativism: we have begun to explain what pure positions are.

Cf. Dedekind's "creation"; "lightweight" ontology (Wright, 1983), (Rayo, 2013)

Cf. the vacillation in (Dedekind, 1888).

Individuation of pure positions: one-by-one

We have not yet defined identity of pure positions **derived from different systems**. Suppose we define:

$$[x]_{\mathcal{S}} = [x']_{\mathcal{S}'} \leftrightarrow \exists f \big(f : \mathcal{S} \cong \mathcal{S}' \wedge f(x) = x' \big)$$

This works well for rigid systems (i.e. no non-trivial automorphisms) (Linnebo and Pettigrew, 2014).

But this definition identifies symmetric positions of any non-rigid structure! Consider the graph G:

$$\begin{tikzcd} v_1 \arrow[r, bend right] & v_2 \arrow[l, bend right] \end{tikzcd}$$

Then $[v_1]_G = [v_2]_G$, since there is an automorphism $f : G \cong G$ such that $f(v_1) = v_2$ and $f(v_2) = v_1$. So $[G] \ncong G$.

Individuation of pure positions: collective

(Litland, 2016) proposes

$$[xx]_{\mathcal{S}} = [yy]_{\mathcal{S}'} \quad \text{:iff} \quad \exists f \big(f : \mathcal{S} \cong \mathcal{S}' \wedge f(xx) = yy \big) \tag{D2}$$

This can be shown to work—provided that xx are closed under automorphisms of $\mathcal{S}$.

To understand the proviso, suppose f is an automorphism of $\mathcal{S}$. Then:

$$\{[x]_{\mathcal{S}} \mid x \prec xx\} = [xx]_{\mathcal{S}} = [f(xx)]_{\mathcal{S}} = \{[f(x)]_{\mathcal{S}} \mid x \prec xx\}$$

Because of the relative individuation of pure positions derived from $\mathcal{S}$, this requires $f(xx) = xx$.

Individuation of pure positions: examples

Suppose $\mathbb{C}$ and $\mathbb{C}'$ are two realizations of the complex field, with imaginary units $\pm i$ and $\pm j$, respectively.

We can individuate the positions of i and $-i$ in $\mathbb{C}$ **collectively but not one-by-one**. We have $[\pm i]_{\mathbb{C}} = [\pm j]_{\mathbb{C}'}$. But there is no fact of the matter as to whether $[i]_{\mathbb{C}} = [j]_{\mathbb{C}'}$.

The entire domain is always closed under automorphisms. So for any $\mathcal{S}$ we have $[D]_S = [D']_{S'}$ whenever $S \cong S'$.

Pure relations: the idea

Consider a relation R on the domain of a system $\mathcal{S}$. We want to define a corresponding pure relation $[R]_{\mathcal{S}}$ on $[D]_{\mathcal{S}}$.

The idea is to let $[R]_{\mathcal{S}}$ hold of some *pure positions* iff R holds of *the occupants* of these positions in $\mathcal{S}$.

So we wish to define:

$$[R]_{\mathcal{S}}([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) \quad :\text{iff} \quad R(a, b) \tag{D3}$$

But we need to be careful . . .

Pure relations: an example

$$\begin{tikzcd}
a \arrow[dr] & & a' \arrow[dl] \\
& b &
\end{tikzcd}$$

By isomorphism invariance:

$$\text{if} \quad f : \mathcal{S} \cong \mathcal{S}' \quad \text{then} \quad [R]_{\mathcal{S}} = [f(R)]_{\mathcal{S}'}$$

Let f be the automorphism swapping a and a'. Then $[R]_{\mathcal{S}} = [f(R)]_{\mathcal{S}}$.

This yields:

$$R(a, b) \quad \text{iff} \quad [R]_S([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) \quad \text{iff} \quad [f(R)]_S([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) \quad \text{iff} \quad f(R)(a, b)$$

Thus, definition (D3) presupposes that R be *invariant*, i.e. that for any automorphism f, we have $f(R) = R$.

Consider a system $\mathcal{S} = \langle D, R_1, \ldots, R_n \rangle$.

1. We posit a domain of pure positions $[D]_{\mathcal{S}} = \{[a]_{\mathcal{S}} \mid a \in D\}$.

2. We ascribe **definable** properties to pure positions **derived from a single system**:

$$\phi([\vec{a}]_{\mathcal{S}}) \quad \text{:iff} \quad \mathcal{S} \models \phi(\vec{a}) \tag{D1}$$

3. A collective individuation of **invariant collections** of pure positions:

$$[xx]_{\mathcal{S}} = [yy]_{\mathcal{S}'} \leftrightarrow \exists f \big(f : \mathcal{S} \cong \mathcal{S}' \wedge f(xx) = yy \big) \tag{D2}$$

4. We define the pure relation derived from any **invariant** relation R:

$$[R]_S([a]_{\mathcal{S}}, [b]_{\mathcal{S}}) :\leftrightarrow R(a, b) \tag{D3}$$

These three definitions ensure $[\mathcal{S}] = [\mathcal{S}']$ iff $\mathcal{S} \cong \mathcal{S}'$, as desired.

The grand unification: purification and abstraction

Recall **Restricted Purity**:

Every "important" relation on a pure system is structural.

Proposal: "important" $=$ inherited

The resulting version of Restricted Purity is

- **true**: $[R]_{\mathcal{S}}$ extends naturally to $\mathcal{X}$ where $f : \mathcal{S} \cong \mathcal{X}$, namely as $f(R)$; i.e. go from relation on pure positions to corresponding relation on occupants of these positions. Cf. (Korbmacher and Schiemer, 2016)
- **non-trivial**: contrast the inherited properties of directions.

The grand unification: abstraction and universals

- Consider the universal of being a system isomorphic to $\mathcal{S}$.
- For each such universal there is a pure system $[\mathcal{S}]$ whose inherited properties are precisely those that are shared by each instance of the universal:

$$\forall \mathcal{X} \forall f \big(f : \mathcal{S} \cong \mathcal{X} \to f(R)(f(\vec{a})) \big) \quad \text{iff} \quad [R]_{\mathcal{S}}([\vec{a}]_{\mathcal{S}})$$

- Thus, $[\mathcal{S}]$ can be seen as a "generic instance" of the universal whose relational facts are all and only those that encode relational facts obtaining in all systems that instantiate the universal.

Concluding remarks

1. Three explications of the idea of a pure structure as a ‘one over many’: purification; types (Frege abstraction); universals.

2. An account of structure abstraction which makes sense of the Purity thesis . . .

3. . . . and brings out the connection with the relevant universals.

The question of reification

- I’ve been non-committal on the question of eliminativism.
- Can we make full sense of the indeterminacy of $[i]_{\mathbb{C}} = [j]_{\mathbb{C}'}$?
- *Semi-robust individuation*: more than just assigning appropriate truth-values to sentences, less than complete individuation.

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