# Structuralism and the Epistemology of Coherence

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Conference on the Foundations of Structuralism, MCMP
12/10/2016

## 1 Structuralism and coherence

A fundamental desideratum in the philosophy of mathematics: provide an epistemology of the subject, i.e. an explanation of how we can form reliable / justified / knowledgable mathematical beliefs. Structuralists, like everyone else, owe such an account.

Can much be said in general, given the diversity of structuralist views? A remarkable convergence: one of a family of notions – coherence, (logical) possibility, satisfiability, consistency – plays a key role. Two representative examples:

- *Ante rem* structuralism (Shapiro). Systems are pluralities of objects bearing relations to one another; structures are abstract patterns or forms that systems exemplify or instantiate. Mathematics is about "places" in the latter, taken as genuine self-subsisting entities.

  The existence of structures is tied very closely to the *coherence* of theories, where coherence is taken to be a "primitive, intuitive", logico-mathematical notion (distinct from proof-theoretic consistency – $PA_2 + \neg Con PA_2$ is consistent but not coherent – and similar in spirit, but not reducible to, set-theoretic satisfiability).

  **Coherence Axiom:** If $\Phi$ is a coherent formula in a second-order language, then there is a structure that satisfies $\Phi$

  This claim does epistemic as well as metaphysical work for Shapiro: our justification for believing in the existence of structures depends, in part, on our having justification to believe in the coherence of theories characterizing them.

- *In re* structuralism (Hellman). Mathematical claims are systematically reinterpreted in terms of a notion of logico-mathematical possibility (Hellman suggests it satisfies S5). For instance, an arithmetical claim $\phi$ is interpreted (roughly) as "necessarily, in any $\omega$-sequence, $\phi$".

  Problem of vacuity: what if $\omega$-sequences (or more generally, systems realizing a given structure) are impossible? Response: adopt as a "basic thesis" of modal mathematics the corresponding possibility claim. So for arithmetic we have:

  **Modal Existence Thesis:** $\Diamond \exists X \exists f (PA_2(X, f))$

Thus for both Hellman and Shapiro, a coherence-like notion is implicated in the epistemology of even the most basic mathematical claims. (I'll assume they are gesturing at the same notion and call it plain old "coherence"). On both pictures, justification in mathematical claims seems to require justification that the background theory is coherent.

So if structuralism is right, then the fundamental question in the epistemology of mathematics is less one of explaining our epistemic access to abstract objects and more one of explaining how we can get justification or knowledge that our best theories *are* in fact coherent.

How might this be done?



## 2 Two responses unavailable to structuralists

There are a number of potentially attractive routes to justification that seem to be unavailable to structuralists here. Can we justify coherence by appeal to...

**...Set Theory?**

It's natural to think that we might show a theory is coherent by showing that it possesses a model in the universe of sets. Indeed, this may work. But it relies on either a background set theoretic ontology or at least prior justification that *set theory* is coherent; so the question is pushed back a step.

**...Truth?**

Could "semantic arguments" for consistency help? They run roughly as follows: the axioms of $T$ are true, the inference rules (of logic) preserve truth, so all of the theorems of $T$ are true. But (pick your favourite falsehood, e.g.) $0 = 1$ is not true, so it's not a theorem, so $T$ is consistent. This argument can be formalized in a theory that extends $T$ by adding suitable compositional axioms for truth; the canonical consistency sentence for $T$ then follows as a theorem.

Three problems: (i) *In re* structuralists will balk at the premises, which involve the (literal) truth of platonistic-sounding mathematical claims. (ii) *Ante rem* structuralists will find the argument problematically circular: it will fail to transmit justification, because to accept the axioms we need prior justification in the conclusion. (iii) The gap between (proof-theoretic consistency) and the desired result (coherence). Further argument seems to be required.

## 3 Ampliative arguments

What about the prospects for non-deductive justification of various kinds?

Enumerative induction is a first thought. Here's a proof in our theory that doesn't have a contradiction as its conclusion, here's another one,... etc etc; so our theory doesn't prove *any* contradiction!

First worry: it's arguable whether enumerative induction in mathematics ever gives us justification at all (as opposed to e.g. motivation for hypotheses). The problem is that the sample is biased because all of the observed cases are *small*. At best the justification provided is pretty weak.

Second worry: it doesn't seem to provide us with resources sufficiently discriminating to say what we intuitively want to say about the strength of justification in different theories. Consider Quine's New Foundations (NF):

(Extensionality) $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$

(Comprehension Schema) $\forall x \exists y (x \in y \leftrightarrow \phi(x))$ where $\phi$ is *stratified* (stratification is a technical, not-very-perspicuous syntactic condition imported from typed set theory).

I think it's fair to say that the consensus is that we don't have justification for thinking that NF is consistent. The main reasons:

- NF has very strange features: there exists a set of all sets, Cantor's theorem ($|\mathcal{P}(X)| > |X|$) fails, the Axiom of Choice is refutable;
- NF doesn't appear to have any "conceptual motivation";
- NF is not known to be equi-consistent with any mainstream mathematical theory.

But the enumerative arguments for our best theories (e.g. ZFC) and NF seem roughly on a par; so there must be more to the epistemological story. (Maybe you think that the enumerative basis is weaker; but (a) this is not obvious in light of how much mathematics can be formalized in NF, and (b) intuitively, just running through a bunch more proofs wouldn't help very much).

Third worry: how do we move from consistency to coherence, again?

A different approach: inference to the best explanation. Two interesting arguments here, depending on what we take the data in question to be.

**IBE Argument #1**: lack of demonstrated inconsistency. We haven't been able to find an inconsistency in our best theories, despite having a reasonably extensive catalogue of tricks used to draw out contradictions and despite expending considerable effort; the best explanation for this is that they're coherent. This doesn't seem like a bad argument; still, some residual dissatisfaction.

First worry: what's so much better about the explanation that the theories are coherent (as opposed to that they incorporate no *small* inconsistency, or no inconsistency that's relatively easy to figure out)? It's surprisingly hard to get a firm answer from the canons of IBE (such as they are).

Second worry: the point about NF recurs, for NF seems similarly resistant to yielding a contradiction in any of the well-known ways.

**IBE Argument #2**: the applicability of mathematics. On this line, the best explanation of the applicability of our theories is their coherence. This argument is interestingly analogous to indispensibility arguments for the *truth* of mathematics; yet nominalist critics of the indispensibility argument may wish to endorse it. Like the last one, this argument seems to have some force. (There's an interesting question about whether it extends to NF). Still, some reasons for dissatisfaction:

First worry: is it really the case that the best explanation of the applicability of mathematics is that it's coherent? Similar problems to the previous case: why aren't other putative explanations just as good? After all, we have experience (e.g. Newtonian fluxion theory) of theories that are plainly inconsistent but wildly successful in applications when handled correctly.

Second worry: Feferman's conjecture, that "surprisingly meager" systems suffice for the formalization of almost all, if not all, scientifically applicable mathematics. Why isn't the best explanation of applicability not the coherence of our best theories *tout court*, but the coherence of the (perhaps fairly weak) fragments used in applications?

These arguments are fine as far as they go, but don't appear to fully track our convictions: even if successful, the support they provide is too weak to make sense of the (absolute and relative) strength of our coherence judgements.

## 4 Conceivability as a guide to coherence

Last proposal I'll sketch: justification in a theory's coherence comes from *possessing a conception* of a structure satisfying its axioms. I think a thought of this kind *psychologically* explains much of the confidence we have. I'll suggest that it's actually more appealing than it might first appear when rationally reconstructed.

What is this – seemingly mysterious – ability to possess a conception of a structure? I suspect the most promising way to proceed assimilates it to the faculty of imagination: we can imagine scenarios in which various mathematical axioms are satisfied, and thereby get justification for thinking that the relevant theory is coherent.

Two major questions arise: can we really conceive of the kinds of infinite scenarios required by serious mathematical theories? And how do we get from conceivability to coherence?

**Conceiving the infinite**

Isn't it blatantly impossible to conceive of infinite scenarios of the kind involved in mathematical theories? Not at all obviously. Exercise: imagine an extremely flat card that's red on one side and green on the other. We need to distinguish between:

(i) visual *images* – transient items of visual experience of specific phenomenological type



(ii) visual *category specifications* – roughly, series of visual images related to one another by continuous changes, such as in viewpoint, orientation, distance, perspective, etc.

Conjecture: we can have visual category specifications of infinite scenarios. For instance, we can imagine a number line (and hence of a model for arithmetic): NB this includes dispositions to "fill in" the imaginative scenario by e.g. continuing indefinitely to add objects to the line as our view of it "pans" rightwards.

**From conceivability to coherence**

Distinguish different notions of conceivability: Ideal ("conceivable on ideal rational reflection") vs prima facie ("conceivable on first appearances"); Positive ("one can form some sort of positive conception of a situation in which S is the case") vs negative ("S is conceivable when S is not ruled out"). What we have in the good cases of imagining scenarios is *prima facie* positive conceivability.

The link between conceivability and possibility is notoriously murky. Some have argued that ideal positive conceivability entails *metaphysical* possibility. The present view doesn't need anything that strong; all it needs is that prima facie positive conceivability gives us *defeasible* justification for coherence (a notion closer to *logical* possibility).

## 5 Conclusions

The conceivability-based account has some appealing features. It gives us an account of a central kind of mathematical justification in a way that doesn't require postulation of mysterious faculties (imagination is presumably fairly secure). It doesn't obviously suffer from the problems faced by the other proposals: no epistemic circularity, and the NF issue is handled nicely by our lack of a conception of an NF-structure. It vindicates the importance placed on possessing a conception of mathematical domains in theorizing (see e.g. the work done on the iterative conception of set).

Some tentative lessons for structuralists, if what I've said is along the right lines:

(1) A notion of coherence (or similar) is fundamental to the epistemology of mathematics, and deserves considerable attention. Structuralists face a serious challenge in explaining our ability to obtain justification in coherence.

(2) Structuralists may very well have to be resourceful in looking for sources of this justification; in particular, something like imagination or mathematical intuition could play a key role.

(3) Suppose (a big supposition!) that we can conceive of models of set theory. This might provide a rationale, on *epistemic* grounds, for structuralists to endorse the tendency to look for "set theoretic foundations" in mathematics.