

On Shelah's L^1_{κ} logic and chain logic

For a cardinal of the form $\kappa = \beth_{\kappa}$, Shelah's rather complex logic $L_{1\kappa}$ has a characterisation as the maximal above $U_{\lambda < \kappa} L_{\lambda, \omega}$ to satisfy the strong undefinability of well order (SUDWO) and Interpolation. SUDWO is a technical strengthening of UDWO. In joint work with Jouko Vaananen, we prove that in the case of such κ being singular of countable cofinality, Karp's chain logic is strictly above $L_{1\kappa}$, while it is already known that it satisfies the undefinability of well order (UDWO) and Interpolation. We also answer a couple more questions that were asked in the literature. Firstly, Problem 1.4. from Shelah 797, asked if for κ singular of countable cofinality there was a logic strictly between $U_{\lambda < \kappa} L_{\lambda, \omega}$ and $U_{\lambda < \kappa} L_{\lambda, \lambda}$ which has Interpolation. While Shelah showed that that $L_{1\kappa}$ satisfies these properties, we show that modulo increasing the upper bound to a model class of $L_{\kappa, \kappa}$, the chain logic, already satisfies the properties. Secondly, we show that the chain logic is not κ -compact, a question that was asked on various occasions. We show some maximality properties of chain logic with respect to UDWO. In conclusion, we show that for all practical purposes the simply defined chain logic emulates the logic $L_{1\kappa}$.