Basic set theory

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1 A set and its members

- A *set* is a collection of objects.
- The objects in a set are called its *elements* or *members*.
- If x is a set and z is a member of x, then we write $z \in x$.
- If x is a set and z is *not* a member of x, then we write $z \notin x$.
- If x and y are sets and if all members of x are also members of y, then we say that x is a subset of y. We write this $x \subseteq y$. Thus, we have:

  $$x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$$

- The members of a set determine that set. That is, there are no two distinct sets with the same members. This distinguishes sets from properties, concepts, and other intensional notions. This is codified in the following axiom of set theory:

  Axiom 1 (Axiom of Extensionality)

  $$\forall x \forall y[x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y)]$$

  Another way of stating this is that if $x \subseteq y$ and $y \subseteq x$, then $x = y$.

  This axiom is reasonably uncontroversial and it is shared by nearly all versions of set theory.

2 Which sets are there?

2.1 Naïve set theory

Naïve Set Theory has two axioms:

- The Axiom of Extensionality.
- The Axiom of Unrestricted Comprehension. This says that, for any property, there is a set of all and only those things that have that property. More precisely, we restrict to properties that can be defined by formulae in the language of set theory with parameters:

Axiom 2 (Axiom of Unrestricted Comprehension) *Given a formula* $\Phi(x, w_1, \ldots, w_n)$ *in the language of set theory,*

$$\forall w_1, \ldots, w_n \exists x \forall z[z \in x \leftrightarrow \Phi(z, w_1, \ldots, w_n)]$$

Given a formula $\Phi(x, w_1, \ldots, w_n)$ and values w_1, …, w_n, we write this set $\{x : \Phi(x, w_1, \ldots, w_n)\}$. This axiom is a modern rendering of Frege's Basic Law V.

Note that this is in fact an *axiom schema*. That is, it is an infinite family of axioms, one for each formula in the language of set theory.

Unfortunately:

Theorem 2.1 (Cantor, Russell) *Naïve Set Theory is inconsistent.*

Proof. Consider the property of being a non-self-membered set: $x \notin x$. Then, by Unrestricted Comprehension, there is a set of all and only the things that have this property:

$$\exists x \forall z[z \in x \leftrightarrow z \notin z]$$

Call this set R. Then we can prove:

$$R \in R \leftrightarrow R \notin R$$

which gives a contradiction. Try it! □

It follows that Frege's Basic Law V is inconsistent.

2.2 Zermelo set theory

Zermelo set theory has seven axioms. But they divide into three types:

- The first are *unconditional existential assertions.* These simply state outright that certain sets exist. They are the Axioms of Empty Set and Infinity.
- The second are *conditional existential assertions.* These state that, whenever a set or some sets exist, then certain other sets exist. They are the Axioms of Pair Set, Union, Power Set, and Separation.
- The third are *structural assertions.* These state something about the internal structure of any given set. These are the Axioms of Extensionality and Foundation.

2.2.1 The Axiom of the Empty Set

This says that there is a set with no elements.

Axiom 3 (Axiom of the Empty Set)

$$\exists x \forall z[z \notin x]$$

By Extensionality, there is at most one such set, so there is a unique empty set. We write it $\varnothing$.

2.2.2 The Axiom of Pair Set

This says that, whenever we have objects x and y, there is a set whose only members are x and y.

Axiom 4 (Axiom of Pair Set)

$$\forall x \forall y \exists u \forall z [z \in u \leftrightarrow z = x \vee z = y]$$

Again, by Extensionality, there is at most one such set, so there is a unique such set. We write it $\{x, y\}$. If $x = y$, we write it $\{x\}$ or $\{y\}$.

2.2.3 The Axiom of Union

This says that, whenever we have a set whose members are further sets, there is a set whose members are precisely the members of those further sets.

Axiom 5 (Axiom of Union)

$$\forall x \exists y \forall z [z \in y \leftrightarrow \exists u \in x (z \in u)]$$

We write this set $\bigcup x$.

Together with the pair set axiom, this allows us to take the union of two sets. Thus, if x and y are sets, we let $x \cup y := \bigcup \{x, y\}$. Thus, the elements of $x \cup y$ are precisely those things that belong either to x or to y or to both.

When we combine this with Pair Set, we can get, for any finite sequence x_1, ..., x_n, the set

$$\{x_1, \ldots, x_n\}$$

2.2.4 The Axiom of the Power Set

This says that, for any set, there is a further set whose members are the subsets of the original set.

Axiom 6 (Axiom of the Power Set)

$$\forall x \exists y \forall z [z \in y \leftrightarrow z \subseteq x]$$

We write this $\mathcal{P}(x)$.

2.2.5 The Axiom of Unrestricted Subset Comprehension

Naïve Set Theory failed because it allowed us to form sets that comprehend arbitrary properties. In Zermelo Set Theory, we are allowed to form subsets that comprehend any property definable by a formulae in the language of set theory with parameters:

Axiom 7 (Axiom of Unrestricted Subset Comprehension) *Suppose* $\Phi(x, w_1, \ldots, w_n)$ *is a formula in the language of set theory. Then*

$$\forall w_1, \ldots, w_n \forall x \exists y \forall z [z \in y \leftrightarrow z \in x \ \& \ \Phi(z, w_1, \ldots, w_n)]$$

We write this set $\{z \in x : \Phi(z)\}$.

Note that, if we treat set theory as a formal system of axioms, the Axiom of Unrestricted Subset Comprehension is, like the Axiom of Unrestricted Comprehension, and *axiom schema*. That is, it is an infinitely family of axioms, one for each formula Φ in the language of set theory.
Thus, for any set x, we are able to form the set of all members of x that are not members of themselves $\{z \in x : z \notin z\}$. Call this set R'. Then we might think that we can derive

$$R' \in R' \leftrightarrow R' \notin R'$$

as in the proof of the inconsistency of Naïve Set Theory. We can certainly prove

$$R' \in R' \rightarrow R' \notin R'$$

but we can't prove

$$R' \notin R' \rightarrow R' \in R'$$

since we would need to know that $R' \in x$ in order for the proof to go through.

Unrestricted Subset Comprehension allows us to define some other set-theoretical constructions that are unavailable without it:

- Given a set x, $\bigcap x := \{z \in \bigcup x : (\forall u \in x)(z \in u)\}$.
- Given sets x and y, $x \cap y := \bigcap\{x, y\}$.
- Given sets x and y, $x - y = \{z \in x : z \notin y\}$.

2.2.6 The Axiom of Infinity

The axioms we have stated so far demand that there be infinitely many sets. For instance, the following sequence of sets is infinite:

$$\varnothing, \quad \mathcal{P}(\varnothing), \quad \mathcal{P}(\mathcal{P}(\varnothing)), \quad \ldots$$

It is generated from $\varnothing$ by the function $x \mapsto \mathcal{P}(x)$. But the axioms do not demand the existence of a set that contains infinitely many elements. We need such a set in order to provide a model for arithmetic and real analysis. So we posit one. In particular, we posit a set that contains all the sets in the infinite sequence just described:

Axiom 8 (The Axiom of Infinity)

$$\exists x[\varnothing \in x \ \& \ \forall z(z \in x \rightarrow \mathcal{P}(z) \in x)]$$

2.2.7 The Axiom of Foundation

So far in our axioms, we have done nothing that rules out the possibility of a set x such $x \in x$, or sets x and y such that $x \in y \in x$. Foundation (also known as Regularity) does this.

Axiom 9 (The Axiom of Foundation)

$$\forall x[x \neq \varnothing \rightarrow \exists z(z \in x \ \& \ z \cap x = \varnothing)]$$

It follows from this that, for any x, $x \notin x$. Suppose $x \in x$. Then consider $\{x\}$. Then, by Foundation, there is $z \in \{x\}$ such that $z \cap \{x\} = \varnothing$. That is, $x \cap \{x\} = \varnothing$. But $x \in x$ (by hypothesis) and $x \in \{x\}$ (by definition). So we have a contradiction. Thus, $x \notin x$.

Definition 2.2 (Z) *The theory with the Axioms of Extensionality, Empty Set, Pair Set, Union, Power Set, Unrestricted Subset Comprehension, Infinity, and Foundation is called* Zermelo set theory *and written Z.*

2.3 Zermelo-Fraenkel Set Theory

In 1922, Fraenkel introduced a further axiom for set theory. This is the Axiom of Replacement. The intuitive idea is this: Suppose we have a set x and we have a function f that takes an element $z \in x$ and returns a value $f(z)$. Then, intuitively, there is a set whose elements are all the values that f takes on elements of x. That is, there is a set $\{y : (\exists z \in x)(f(z) = y)\}$. This is called the *range of f*. Replacement essentially says that the range of f exists. The problem is that we don't yet have a way of talking about functions in set theory. And, when we do give the standard definition as sets of ordered pairs, it requires that we already know that the function has a range. Thus, we need to proceed in another way.

Note that functions are relations of a special sort. And relations are defined by formulae in the language of set theory with two free variables. We say that a formula $\Phi(x, y, w_1, \ldots, w_n)$ defines a function on a set u relative to parameters $w_1, \ldots, w_n$, if, for all $x \in u$, there is a unique y such that $\Phi(x, y, w_1, \ldots, w_n)$. Thus, we state replacement as follows:

Axiom 10 (Axiom of Replacement) *Suppose $\Phi(x, y, w_1, \ldots, w_n)$ is a formula in the language of set theory. Then*

$$\forall w_1, \ldots, w_n \forall u[(\forall x \in u)(\exists! y)\Phi(x, y)) \to \exists w \forall y[y \in w \leftrightarrow (\exists x \in u)\Phi(x, y)]]$$

Definition 2.3 (Z) *The theory with the Axioms of Extensionality, Empty Set, Pair Set, Union, Power Set, Unrestricted Subset Comprehension, Infinity, Foundation, and Replacement is called* Zermelo-Fraenkel set theory *and written ZF.*

3 Some set-theoretic constructions

3.1 Ordered pairs

We often talk about the ordered pair (a, b). But what is it? We want to give a set that (a, b) names such that the following holds:

$$(a, b) = (c, d) \text{ iff } a = c \,\&\, b = d$$

Here is the Kuratowski definition:

$$(a, b) := \{\{a\}, \{a, b\}\}$$

This is not the only satisfactory definition, but there is a sense in which it is the simplest. Notice that its existence follows from the Axiom of Pair Set alone.

A similar trick gives us ordered triples:

$$(a, b, c) := \{\{a\}, \{a, b\}, \{a, b, c\}\}$$

And in general:

$$(a_1, \ldots, a_n) := \{\{a_1\}, \{a_1, a_2\}, \ldots, \{a_1, \ldots, a_{n-1}\}, \{a_1, \ldots, a_n\}\}$$

3.2 Cartesian product

Sometimes, given two sets x and y, we want to talk about the set of ordered pairs (a, b) such that $a \in x$ and $b \in y$. We call this the *Cartesian product* of x and y and write it $x \times y$:

$$x \times y := \{(a, b) : a \in x \;\&\; b \in y\}$$

Notice that the existence of this set is guaranteed by the Axioms of Power Set and Union.

Similarly, we have:

$$x_1 \times \ldots \times x_n := \{(a_1, \ldots, a_n) : a_1 \in x_1, \ldots, a_n \in x_n\}$$

Given a number m, we write:

$$x^m = \underbrace{x \times \ldots \times x}_{m\,\text{times}}$$

3.3 Relations

- An m-place relation over a set x is a subset of x^m: that is, it is a set of m-tuples whose elements lie in S.
- We have names for various types of two-place (or *binary*) relation. Suppose R is a two-place relation over x:
  - R is *reflexive* iff $(\forall z \in x)Rzz$
  - R is *symmetric* iff $(\forall y, z \in x)(Ryz \supset Rzy)$.
  - R is *transitive* iff $(\forall u, v, w \in x)((Ruv \;\&\; Rvw) \supset Ruw)$
- We say that a binary relation R is an *equivalence relation* if R is reflexive, symmetric, and transitive.
- Given an equivalence relation R over a set x, and given $z \in x$, we define the *equivalence class of z* as follows:
$$[z] = \{u \in x : Ruz\}$$
- It is a theorem that, if R is an equivalence relation over x, then the equivalence classes of R *partitions* x into mutually exclusive subsets. That is:
  - For all $z \in x$, there is $u \in x$ (namely, z itself) such that $z \in [u]$; *and*
  - For all $y, z \in x$ either $[y] = [z]$ or $[y] \cap [z] = \varnothing$.