Bayesian epistemology II: Arguments for Probabilism

Richard Pettigrew

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1 The model

Represent an agent's credal state at a given time t by a *credence function*

$$c_t : \mathcal{F} \to [0, 1].$$

where $\mathcal{F}$ is the algebra of propositions about which the agent has an opinion.[1]

- If $A \in \mathcal{F}$, then $c_t(A) = 0$ iff the agent has minimal credence in A at t.
- If $A \in \mathcal{F}$, then $c_t(A) = 1$ iff the agent has maximal credence in A at t.

Note: It is an empirical assumption that agents are capable of maximal and minimal credences; it is not a normative claim.

2 The norms

At any time t in her epistemic life, an agent ought to have a credence function c_t such that

- **Probabilism** c_t is a probability function.

  That is,

  - $c_t(\bot) = 0$ and $c_t(\top) = 1$.
  - $c_t(A \vee B) = c_t(A) + c_t(B)$ if A and B are mutually exclusive.

- **Countable additivity** c_t is countably additive.

  That is, if $\mathcal{F}$ is infinite,

  - $c_t(\bigcup_n A_n) = \sum_n c_t(A_n)$ if $A_1, A_2, \ldots$ are pairwise mutually exclusive.

[1] Since $\mathcal{F}$ is an algebra, it is closed under conjunctions, disjunctions, and negations.

3 Dutch Book arguments

3.1 Probabilism

We will assume that $\mathcal{F}$ is finite. This assumption is not necessary, but it simplifies proofs.

We begin by giving an alternative formulation of **Probabilism**.

Definition 1 *An* assignment of truth values to the propositions in $\mathcal{F}$ *is a function* $v : \mathcal{F} \to \{0, 1\}$ *such that*

$$v(\neg A) = \begin{cases} 0 & \textit{if } v(A) = 1 \\ 1 & \textit{if } v(A) = 0 \end{cases}$$

and

$$v(A \vee B) = \begin{cases} 0 & \textit{if } v(A) = 0 \textit{ and } v(B) = 0 \\ 1 & \textit{otherwise} \end{cases}$$

Definition 2 *Let $\mathcal{V}$ be the set of all assignments of truth values to propositions in $\mathcal{F}$.*

We might think of each assignment of truth values as a possible world. Thus, $\mathcal{V}$ is the set of all possible worlds. Note that since $\mathcal{F}$ is finite, $\mathcal{V}$ is finite.

Definition 3 *Let $\mathcal{V}^+$ be the convex hull of $\mathcal{V}$. That is,*

$$\mathcal{V}^+ := \left\{ \sum_{v \in \mathcal{V}} \lambda_v v : \lambda_v > 0 \textit{ and } \sum_{v \in \mathcal{V}} \lambda = 1 \right\}$$

Another characterization of $\mathcal{V}^+$: it is the smallest convex set that contains all elements of $\mathcal{V}$.

Lemma 1 *An agent satisfies **Probabilism** iff her credence function c_t is in $\mathcal{V}^+$.*

Proof. Suppose $c_t \in \mathcal{V}^+$. To see that c_t is a probability function, it suffices to note that:

(i) Each $v \in \mathcal{V}$ is a probability function.

(ii) If p and p' are probability functions, then $\lambda p + (1 - \lambda)p'$ is a probability function.

For the converse, suppose that c_t is a probability function. Then, for each $v \in \mathcal{V}$, let

$$A_v := \bigwedge_{v(A)=0} \neg A \wedge \bigwedge_{v(A)=1} A$$

Thus, A_v is the unique proposition in $\mathcal{F}$ such that

$$v'(A_v) = \begin{cases} 0 & \text{if } v \neq v' \\ 1 & \text{if } v = v' \end{cases}$$

That is, A_v is made true by v but by no other truth assignment. Thus:

$$A = \bigvee_{v(A)=1} A_v$$

And the A_vs are disjoint.

Now let

$$\lambda_v := c_t(A_v)$$

Then, since c_t is a probability function,

$$c_t(A) = c_t(\bigvee_{v(A)=1} A_v) = \sum_{v(A)=1} c_t(A_v) = \sum_{v \in \mathcal{V}} v(A) c_t(A_v) = \sum_{v \in \mathcal{V}} \lambda_v v(A)$$

as required. $\square$

So far, we have been treating credence functions and truth value assignments as functions from $\mathcal{F}$ into $[0, 1]$. But, if $\mathcal{F} = \{A_1, \ldots, A_n\}$, then we might just as well consider them as vectors in an n-dimensional vector space. Thus, if c is a credence function, we represent it as

$$c = (c_1, \ldots, c_n)$$

where $c_i = c(A_i)$. Similarly, if v is a truth value assignment, we represent it as

$$v = (v_1, \ldots, v_n)$$

where $v_i = v(A_i)$. Using this notation, we can better state the Dutch Book argument.

A *Dutch book* is a book of bets on the propositions in $\mathcal{F}$ and a price for that book such that the price is greater than the payoff of the book of bets in every possible world.

We represent this mathematically as follows:

- Then a book of bets on $\mathcal{F}$ is represented by a vector $(s_1, \ldots, s_n)$. This is a set of n bets $B_1, \ldots, B_n$, where:
  - B_i will pay £s_i if A_i is true;
  - B_i will pay £0 if A_i is false.
- Suppose p_i is the price for bet B_i. Then the price of the book $(s_1, \ldots, s_n)$ is $\sum_i p_i$.
- The payoff of the book $B = (s_1, \ldots, s_n)$ at $v \in \mathcal{V}$ is

  $$\sum_i v_i s_i$$

- Thus, a book $(s_1, \ldots, s_n)$ with prices p_i for bet B_i is a Dutch Book iff

  $$\sum_i p_i > \sum_i v_i s_i$$

  for all $v \in \mathcal{V}$.

3.1.1 The argument

(1) **Credences as betting odds** An agent with credence r in proposition A should consider £rS as a fair price for a bet that pays £S if A is true and £0 if A is false.

(2) **Package principle** If an agent considers £r as a fair price for bet 1 and £r' as a fair price for bet 2, then she should consider £$(r + r')$ as a fair price for the book of bets that consists of bets 1 and 2.

(3) **Undutchbookable** An agent should not have credences that lead her to accepting a Dutch book as fair.

(3) **Theorem 1 (Dutch book theorem)**

(I) *If $c \notin \mathcal{V}^+$, then there is a book of bets $(s_1, \ldots, s_n)$ such that, for all $v \in \mathcal{V}$,*

$$\sum_i c_i s_i > \sum_i v_i s_i$$

(II) *If $c \in \mathcal{V}^+$, then there is no book of bets $(s_1, \ldots, s_n)$ such that, for all $v \in \mathcal{V}$,*

$$\sum_i c_i s_i > \sum_i v_i s_i$$

Therefore,

(4) An agent ought to obey **Probabilism**.

3.1.2 Proof of Theorem 1

(I) Suppose $c \notin \mathcal{V}^+$. Then let $p \in \mathcal{V}^+$ be the point in $\mathcal{V}^+$ that is closest to c. Then let $s = c - p$. Then, by a classical geometrical result, we have that the angle between s and $v - p$ is not acute, for any $v \in \mathcal{V}$. Thus $s \cdot (v - p) \leq 0$. This gives $s \cdot v \leq s \cdot p$. But we also have $||s||^2 > 0$. But

$$||s||^2 = s \cdot s = s \cdot (c - p) = s \cdot c - s \cdot p$$

Thus, $s \cdot p < s \cdot c$. So $s \cdot v < s \cdot c$. That is,

$$\sum_i c_i s_i > \sum_i v_i s_i$$

as required.

(II) Suppose $c \in \mathcal{V}^+$. And let s be a vector. Then, *either* the angle between $v - c$ and s is right for all $v \in \mathcal{V}$ *or* for at least one $v \in \mathcal{V}$, the angle between s and $v - c$ is acute.

- If the angle between s and $v - c$ is right for all $v \in \mathcal{V}$, then $s \cdot (v - c) = 0$, so

$$\sum_i c_i s_i = \sum_i v_i s_i$$

for all $v \in \mathcal{V}$.

- If the angle between s and $v - c$ is obtuse, then $s \cdot (v - c) > 0$, so

$$\sum_i c_i s_i < \sum_i v_i s_i$$

This completes the proof. □

4 Accuracy domination arguments

In accuracy domination arguments, we treat epistemic states as epistemic acts, we introduce measures of epistemic utility for those acts, and we employ the machinery of decision theory to derive norms that govern epistemic states.

An epistemic utility argument requires:

- **An epistemic utility function**

  For each credence function c and possible world w, $EU(c, w)$ measures the epistemic goodness of having c at w.

  An example: The *Brier score* is the following measure:

  $$B(c, w) = 1 - \sum_{A \in \mathcal{F}} (c(A) - v_w(A))^2$$

  where

  $$v_w(A) = \begin{cases} 0 & \text{if } A \text{ is false} \\ 1 & \text{if } A \text{ is true} \end{cases}$$

- **A norm of decision theory**

  This tells us how an agent should choose from a range of different acts on the basis of the epistemic utility of those acts at different worlds. Example:

  > **Act-Type Dominance** Suppose there are two sorts of act: $\mathcal{D}_1$ and $\mathcal{D}_2$. Say that an act D is dominated by another act D' if D' has higher utility than D in every world. Now suppose:
  > – Every D in $\mathcal{D}_1$ is dominated by some D' in $\mathcal{D}_2$.
  > – No D in $\mathcal{D}_2$ is dominated by any D' in $\mathcal{D}_1$ or in $\mathcal{D}_2$.
  >
  > Then the agent should choose an act from $\mathcal{D}_2$.

4.1 The argument

(1) **The Brier score measures epistemic utility**

(2) **Act-Type Dominance**

(3) **Theorem 2 (de Finetti)**

(I) *If $c \notin \mathcal{V}^+$, then there is $p \in \mathcal{V}^+$ such that*

$$B(c, w) < B(p, w)$$

for all worlds w.

(II) *If $c \in \mathcal{V}^+$, then there is no credence function p such that*

$$B(c, w) < B(p, w)$$

for all worlds w.

Therefore,

(4) An agent ought to obey **Probabilism**.