Bayesian epistemology III: Arguments for Conditionalization

Richard Pettigrew

May 10, 2012

1 The model

- Let $\mathcal{F}$ be the algebra of propositions about which the agent has an opinion.
- Represent an agent's credal state at a given time t by a *credence function*
$$c_t : \mathcal{F} \to [0, 1].$$
- Represent an agent's total evidence at a given time t by a proposition E_t.

2 The norms

Bayesian Conditionalization For any two times $t' > t$ in an agent's epistemic life, an agent ought to have credence functions c_t and $c_{t'}$ such that
$$c_{t'}(A) = c_t(A|E_{t'})$$

3 A pragmatic argument

The original version is due to Peter M. Brown [Brown, 1976]. Throughout, we assume that c_t and $c_{t'}$ are probability functions.

- Let $\mathcal{E}_{t'} = \{E_1, \ldots, E_n\}$ be a partition. It gives the propositions that our agent might learn by t'.
- Given $E_i \in \mathcal{E}$, let A_i be the set of actions that will be open to the agent if E_i is true.
- Let U be a utility function that takes each $E_i \in \mathcal{E}_{t'}$, each action $a \in A_i$, and each world $w \in E_i$ and returns a measure $U(a, w)$ of the utility of the outcome of a at w.
- Given a credence function c, let $A_i(c)$ be an action from A_i that maximizes expected utility relative to c and in the presence of evidence E_i.

  That is, for all $a \in A_i$,
$$\sum_{w \in E_i} c(w)U(A_i(c), w) \geq \sum_{w \in E_i} c(w)U(a, w)$$

That is, $A_i(c)$ is the action that a rational agent will choose at t' if she has learned E_i and if her credence function at t' is c.

- An updating rule is a function R that takes a credence function c and a piece of evidence $E_i \in \mathcal{E}$ and returns a credence function $R_c(E_i)$.
- For instance, the conditionalization rule Cond is defined as follows:

$$\mathrm{Cond}_c(E_i) = c(\cdot|E_i)$$

- Then we define the utility of adopting an updating rule R when one has credence function c at a world $w \in E_i$ as follows:

$$U(R_c, w) = U(A_i(R_c(E_i)), w)$$

With this terminology in hand, the pragmatic argument for conditionalization goes as follows:

Theorem 1 *The updating rule* Cond *maximizes expected utility relative to any credence function c and any partition* $\mathcal{E} = \{E_1, \ldots, E_n\}$*.*

That is, if R is an updating rule, then

$$\sum_{w \in W} c(w)U(\mathrm{Cond}_c, w) \geq \sum_{w \in W} c(w)U(R_c, w)$$

Proof. First, we have:

$$\sum_{w \in E_i} c(w|E_i)U(A_i(c(\cdot|E_i)), w) \geq \sum_{w \in E_i} c(w|E_i)U(a, w)$$

This is by the definition of $A_i(c(\cdot|E_i))$. In particular, for any updating rule R, we get:

$$\sum_{w \in E_i} c(w|E_i)U(A_i(c(\cdot|E_i)), w) \geq \sum_{w \in E_i} c(w|E_i)U(R_c(E_i), w)$$

And, since $\mathrm{Cond}_c(E_i) = c(\cdot|E_i)$, we get:

$$\sum_{w \in E_i} c(w|E_i)U(A_i(\mathrm{Cond}_c(E_i)), w) \geq \sum_{w \in E_i} c(w|E_i)U(R_c(E_i), w)$$

From this, and the fact that $w \in E_i$, we get:

$$\sum_{w \in E_i} \frac{c(w)}{c(E_i)}U(A_i(\mathrm{Cond}_c(E_i)), w) \geq \sum_{w \in E_i} \frac{c(w)}{c(E_i)}U(R_c(E_i), w)$$

and thus

$$\sum_{w \in E_i} c(w)U(A_i(\mathrm{Cond}_c(E_i)), w) \geq \sum_{w \in E_i} c(w)U(R_c(E_i), w)$$

By the definition of utility for an updating rule, we have:

$$\sum_{w \in E_i} c(w)U(\mathrm{Cond}_c, w) \geq \sum_{w \in E_i} c(w)U(R_c, w)$$

Thus,

$$\sum_{E_i \in \mathcal{E}} \sum_{w \in E_i} c(w)U(\text{Cond}_c, w) \geq \sum_{E_i \in \mathcal{E}} \sum_{w \in E_i} c(w)U(R_c, w)$$

And thus,

$$\sum_{w \in W} c(w)U(\text{Cond}_c, w) \geq \sum_{w \in W} c(w)U(R_c, w)$$

as required. □

4 An epistemic argument

The original version is due to Hilary Greaves and David Wallace [Greaves and Wallace, 2006].

- This time, we let EU be an epistemic utility function. That is, the utility of a credence function is not defined in terms of the utility of actions that it sanctions. Thus, $EU(c, w)$ measures the purely epistemic utility of having credence function c at world w.
- Given an updating rule R and a credence function, we define $EU(R_c, w)$ as follows: if $w \in E_i$,
$$EU(R_c, w) = EU(R_c(E_i), w)$$
- We say that a credence function is *proper* if it expects itself to have greater epistemic utility than it expects any other credence function to have. That is, for any $c \neq c'$,
$$\sum_{w \in W} c(w)EU(c, w) > \sum_{w \in W} c(w)EU(c', w)$$

Theorem 2 *Suppose* $\mathcal{E} = \{E_1, \ldots, E_n\}$ *is a partition. And suppose that each* $\text{Cond}_c(E_i)$ *is proper for each* E_i *relative to EU. Then if* $R_c \neq \text{Cond}_c$*,*

$$\sum_{w \in W} c(w)EU(\text{Cond}_c, w) > \sum_{w \in W} c(w)EU(R_c, w)$$

Proof. The proof is almost identical to the proof in the pragmatic argument. Since $\text{Cond}_c(E_i) = c(\cdot|E_i)$ is proper, we have:

$$\sum_{w \in W} c(w|E_i)EU(c(\cdot|E_i), w) > \sum_{w \in W} c(w|E_i)EU(c', w)$$

if $c' \neq c(\cdot|E_i)$. Thus, for all E_i, we have:

$$\sum_{w \in W} c(w|E_i)EU(c(\cdot|E_i), w) \geq \sum_{w \in W} c(w|E_i)EU(R_c, w)$$

for any updating rule R_c. Furthermore, if $R_c \neq \text{Cond}_c$, then there is E_i such that

$$\sum_{w \in W} c(w|E_i)EU(c(\cdot|E_i), w) > \sum_{w \in W} c(w|E_i)EU(R_c, w)$$

Thus, for all E_i, we have

$$\sum_{w \in E_i} \frac{c(w)}{c(E_i)} EU(c(\cdot|E_i), w) \geq \sum_{w \in E_i} \frac{c(w)}{c(E_i)} EU(R_c, w)$$

since $c(w|E_i) = \frac{c(w)}{c(E_i)}$ if $w \in E_i$ and $c(w|E_i) = 0$ if $w \notin E_i$. Thus,

$$\sum_{w \in E_i} c(w) EU(c(\cdot|E_i), w) \geq \sum_{w \in E_i} c(w) EU(R_c, w)$$

for all E_i with strict inequality for at least one E_i. And so

$$\sum_{E_i \in \mathcal{E}} \sum_{w \in E_i} c(w) EU(c(\cdot|E_i), w) \geq \sum_{E_i \in \mathcal{E}} \sum_{w \in E_i} c(w) EU(R_c, w)$$

for all E_i with strict inequality for at least one E_i. Finally, it follows that,

$$\sum_{w \in W} c(w) EU(c(\cdot|E_i), w) > \sum_{w \in W} c(w) EU(R_c, w)$$

as required. □

References

[Brown, 1976] Brown, P. M. (1976). Conditionalization and Expected Utility. *Philosophy of Science*, 43(3):415–419.

[Greaves and Wallace, 2006] Greaves, H. and Wallace, D. (2006). Justifying Conditionalization: Conditionalization Maximizes Expected Epistemic Utility. *Mind*, 115(459):607–632.