An introduction to toposes

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Contents

Chapter 1

Motivating category theory

These notes are intended to provided a self-contained introduction to the particular sort of category called a *topos*. For this reason, much of the early material will be familiar to those acquainted with the definitions of category theory. The table of contents should give a good idea how far you have to skip ahead to start reading.

For missing proofs, consult the introductory texts by Goldblatt [**?**], Lawvere and Rosebrugh [**?**], or McLarty [**?**].

1.1 The idea behind category theory

The motto of the category theorist might be this:

> Ask not what a thing is; ask what it does.

The things in question are mathematical objects, such as the direct product of two groups, the function space between two sets, the tangent space at a point on a differentiable manifold, or the disjoint union of a two sets. Non-categorical treatments of group theory, topology, differential geometry, and set theory describe these objects by answering the question: What is it? They answer with a description in terms of the intrinsic properties of the object. For instance, the direct product of two groups G and H is defined to be the further group whose underlying set is the set of all ordered pairs (g, h) with $g \in G$ and $h \in H$, and whose binary operation is defined by $(g_1, h_1), (g_2, h_2) \mapsto (g_1 *_G g_2, h_1 *_H h_2)$. Similarly, the disjoint union of two sets A and B is defined to be a further set $(A \times \{0\}) \cup (B \times \{1\})$.

The motivation for category theory is that these definitions in terms of intrinsic properties are more detailed than is required for mathematics and, as a consequence of their superfluous detail, not general enough. The particular definition of the direct product for groups given above is not the only one that will do; nor is the particular definition of the disjoint union of two sets. Rather, there are many different particular definitions that define different particular objects by stating their intrinsic features and are such that any one of these definitions would serve the mathematical purpose just as well as any other. All that is required is that the objects picked out by these particular definitions have certain properties, which can be expressed in terms of certain sorts of functions between mathematical objects of that sort (see, for instance, the definitions of *products* and *coproducts* below, as well as the definition of *intersection* and *union*). This is what the category theorist seeks.

To do this, the category theorist defines a sort of mathematical object called a *category*, which consists of *objects* and *arrows* (sometimes called *mappings* or *morphisms*). The definition is extremely general. For example, any particular group can be seen as a category containing a single object with arrows corresponding to each element of the group.[1] But these particular 'concrete' sorts of mathematical object are not the main focus of category theory. Rather, it is concerned with categories whose objects are *all* particular 'concrete' mathematical objects of a certain sort. Thus, there is a category called Grp, whose objects are the mathematical objects that we call groups and whose arrows are the homomorphisms between groups. And there is a category called Set, whose objects are the mathematical objects we call sets and the functions between them. Granted the category Grp, one can define what one means by a particular group being a direct product to two other groups. One states the definition not in terms of the underlying set of the two given groups, nor in terms of their binary operations, as we did above. Rather, one states it in terms of other groups and arrows between them. Similarly, granted the category Set, one can define what one means by a particular set being a disjoint union of two given sets; and the definition appeals only to sets and the functions between them.

It turns out—though this is something of a bonus, rather than a motivation for category theory—that the definition one gives of a direct product in the category Grp is exactly the same as the definition one gives of a Cartesian product of two sets in the category Set, the definition one gives of the product topology in the category Top of topological spaces and continuous mappings, and (perhaps most surprisingly), the definition of a greatest lower bound in a category based on a partially ordered set $(A, \leq)$. Thus, the same general definitions given by the category theorist turn up in many areas of mathematics. And of course this facilitates the 'lifting' of theorems from one discipline to another.

[1]The binary operation of the group is then represented by the composition of the arrows.

Chapter 2

The definition of a category

So much for motivation. Let us begin on the technical material. A category is an algebraic object like a group or a ring or a field. That is, it consists of an underlying collection of entities along with certain functions and relations on this collection that satisfy certain conditions, which we call the *axioms* of the algebraic object.

In the case of a category, the underlying collection of entities consists of the *objects* and the *arrows* (or *mappings* or *morphisms*) of the category. And there are three functions:

- The *domain* function takes any arrow to an object.
- The *codomain* function takes any arrow to an object.
- The *composition* function takes certain pairs of arrows to arrows.

And the conditions they must satisfy are called the axioms of a category.

Definition 2.0.1 (Category) *A* category $\mathcal{C}$ *consists of*

(i) *a collection* $\mathrm{Ob}(\mathcal{C})$ *of* objects A, B, C, ...

(ii) *a collection* $\mathrm{Ar}(\mathcal{C})$ *of* arrows (*or* morphisms) f, g, h, ...

(iii) *the domain function* $\mathrm{Dom} : \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$

(iv) *the codomain function* $\mathrm{Cod} : \mathrm{Ar}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$

We write $A \xrightarrow{f} B$ *if* $\mathrm{Dom}(f) = A$ *and* $\mathrm{Cod}(f) = B$.

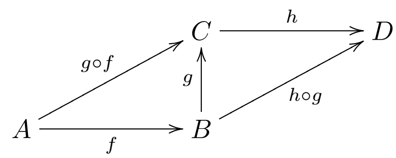
(v) *a composition function* $\circ$ *that takes two arrows* $A \xrightarrow{f} B$ *and* $B \xrightarrow{g} C$ *to a third arrow* $A \xrightarrow{g \circ f} C$ *(read 'g following f' and often written 'gf').*

such that

(1) Associativity of composition *For any three arrows* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

That is, the following diagram commutes:[1]

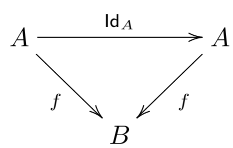


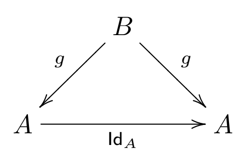
(2) The identity axioms *For every object* A*, there is an* $A \xrightarrow{\mathsf{Id}_A} A$ *such that*

a) *For any* $A \xrightarrow{f} B$, $f \circ \mathsf{Id}_A = f$ *and*

b) *For any* $B \xrightarrow{g} A$, $\mathsf{Id}_A \circ g = g$.

That is, the following diagrams commute:





Example 1 *Here are some examples of categories:*

(1) Set*:*

- *Objects: sets.*
- *Arrows: functions between sets with the domain and codomain specified.*

Thus, in Set, $f : \{a\} \to \{a\}$ *defined by* $f(a) = a$ *is distinct from* $f : \{a\} \to \{a, b\}$ *defined by* $f(a) = a$.

(2) Grp

- *Objects: groups.*
- *Arrows: group homomorphisms.*

(3) *Any partially ordered set (or poset)* $(A, \leq)$ *forms a category:*

- *Objects: the elements of* A

[1] We say that a diagram commutes if, for any two objects A and B in the diagram, any function obtained by following a path in the diagram between A and B, and composing functions as you go, is identical with any other function so obtained.

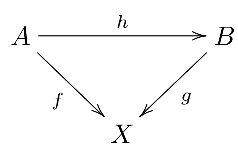
- *Arrows: if $a \leq b$, there is a single arrow between a and b; if $a \not\leq b$, there is no arrow between a and b.*

The efficacy of composition is guaranteed by the transitivity of $\leq$. The associativity of composition *is guaranteed by the uniqueness of arrows between objects.*

The existence of identity arrows is guaranteed by the reflexivity of $\leq$. The identity axioms *are guaranteed by the uniqueness of arrows between objects.*

(4) *Given any category $\mathcal{C}$ and any object X of $\mathcal{C}$, we define the* slice category $\mathcal{C}/X$ of $\mathcal{C}$ over X *as follows:*

- *Objects: all $\mathcal{C}$-arrows $A \xrightarrow{f} X$ with codomain X*
- *Arrows: for each pair of $\mathcal{C}/X$-objects $A \xrightarrow{f} X$ and $B \xrightarrow{g} X$, the $\mathcal{C}/X$-arrows $f \xrightarrow{h} g$ from f to g are the $\mathcal{C}$-arrows $A \xrightarrow{h} B$ from A to B that make the following diagram commute:*

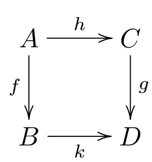


(5) *Given an category $\mathcal{C}$, we define the* arrow category $\mathcal{C}^{\rightarrow}$ of $\mathcal{C}$ *as follows:*

- *Objects: all $\mathcal{C}$-arrow $A \xrightarrow{f} B$.*
- *Arrows: for each pair of $\mathcal{C}^{\rightarrow}$-objects $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, the $\mathcal{C}^{\rightarrow}$-arrows from f to g are the pairs of $\mathcal{C}$-arrows*

$$(A \xrightarrow{h} C, B \xrightarrow{k} D)$$

such that the following diagram commutes:



(6) *Given a category $\mathcal{C}$, we define the category $\mathcal{C}^2$ as follows:*

- *Objects: all pairs (A, B) where A and B are $\mathcal{C}$-objects*
- *Arrows: for each pair of $\mathcal{C}^2$-objects (A_1, A_2) and (B_1, B_2), the $\mathcal{C}^2$-arrows from (A_1, A_2) to (B_1, B_2) are the pairs (f_1, f_2) where f_1 and f_2 are $\mathcal{C}$-arrows such that $A_1 \xrightarrow{f_1} B_1$ and $A_2 \xrightarrow{f_2} B_2$.*

Chapter 3

Monics, epics, isos

In order to state many of the definitions in what follows, we will need to identify some types of arrows:

Definition 3.0.2 *Suppose* $A \xrightarrow{f} B$ *is an arrow of a category* $\mathcal{C}$*. Then*

- f *is* monic *(written* $A \overset{f}{\rightarrowtail} B$*) if*

  For all $\mathcal{C}$*-arrows* $C \overset{g,h}{\rightrightarrows} A$*,* $fg = fh \Rightarrow g = h$*.*[1]

  Such an arrow is also called left-cancellable *or a* monomorphism*.*

- f *is* epic *(written* $A \overset{f}{\twoheadrightarrow} B$*) if*

  For all $\mathcal{C}$*-arrows* $B \overset{g,h}{\rightrightarrows} C$*,* $gf = hf \Rightarrow g = h$*.*

  Such an arrow is also called right-cancellable *or an* epimorphism*.*

- f *is* left-invertible *if*

  There is a $\mathcal{C}$*-arrow* $B \xrightarrow{g} A$ *such that* $gf = \mathsf{Id}_A$*.*

- f *is* right-invertible *if*

  There is a $\mathcal{C}$*-arrow* $B \xrightarrow{g} A$ *such that* $fg = \mathsf{Id}_B$*.*

- f *is* iso *(written* $A \overset{f}{\cong} B$*) if* f *is left- and right-invertible.*

Proposition 3.0.3

(1) *If* f *is left-invertible, then* f *is monic.*

Indeed, a left-invertible arrow is sometimes called a split monic*.*

[1] Here, and for much of what follows, we write 'fg' to mean '$f \circ g$'.

(2) *If f is right-invertible, then f is epic.*

Indeed, a right-invertible arrow is sometimes called a split epic.

(3) *If f is iso, then there is $B \xrightarrow{g} A$ such that $fg = \mathsf{Id}_B$ and $gf = \mathsf{Id}_A$.*

Proof. (1) and (2) are obvious. Suppose $h_L f = \mathsf{Id}_A$ and $f h_R = \mathsf{Id}_B$. Then $h_R = \mathsf{Id}_A h_R = (h_L f) h_R = h_L (f h_R) = h_L \mathsf{Id}_B = h_L$. So $h_L = h_R$ is the required left and right inverse. □

Example 2

- *In* Set*:*
  - *f is monic iff f is injective*
  - *f is epic iff f is surjective*
  - *Every epic is right-invertible iff Choice holds.*

Chapter 4

Pursuing the definition of a topos

In these notes, we are interested in a particular sort of category called a *topos*. The definition of topos was introduced in 1970 by Lawvere and Tierney who were both doctoral students of Eilenberg, the co-founder of category theory with Mac Lane. Their idea was to give a generalization of the concept of set by defining a type of category that has many but not all the features of Set.

More particularly, a topos will satisfy the same *basic existential* axioms as Set, but it might not satisfy the same *higher existential axioms* nor the same *structural* axioms. As we will see, a topos contains analogues of the set-theoretic constructions of disjoint unions, Cartesian products, and function spaces—though, as always in category theory, these are defined by their functional role, rather than by conditions of membership; in fact, membership plays a minor role in the theory of toposes. But a topos might differ from Set in that it might fail to satisfy an analogue of the Axiom of Extensionality, or it might not contain an infinite set, or it might not satisfy the classical laws of logic.

Thus, we will build up our definition of a topos by pursuing categorial definitions of addition, multiplication, and exponentiation of sets—these giving the essential content of the basic existential axioms of set theory—and demand that a topos satisfy these definitions. We will treat addition and multiplication in Chapter 3 and exponentiation in Chapter 8. We will in fact add stronger axioms than are required to guarantee these conditions. This will give the second and third axioms of a topos. We will then consider analogues of subsets of a set in Chapter 9, and we will give our fourth and final axiom. Finally, we will consider different sorts of topos in Chapter 12.

Our first axiom of a topos is, of course:

Axiom 1 *A topos is a category.*

Chapter 5

Diagrams, cones, cocones, limits, colimits

In this section, we introduce the notion of a *diagram* in a category and the notions of a *limit* and a *colimit* over a diagram. These notion have tremendous generality and will provide us with our most elegant definitions.

Definition 5.0.4 (Diagram) *If $\mathcal{C}$ is a category, a* diagram $\mathbf{D}$ in $\mathcal{C}$ *consists of*

(i) *a collection of $\mathcal{C}$-objects;*

(ii) *a collection of $\mathcal{C}$-arrows between objects in this collection.*[1]

Example 3 *Let $\mathcal{C}$ be the following category:*

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$$

Then

$$B \underset{h}{\overset{g}{\rightrightarrows}} C$$

and

$$A \qquad B$$

are examples of diagrams in $\mathcal{C}$.

[1] Officially, a diagram in $\mathcal{C}$ is a functor from a category $\mathcal{J}$ into $\mathcal{C}$. But we will not meet functors until later, so we keep the definition unofficial for the time being.

5.1 Cones, Limits, and Products

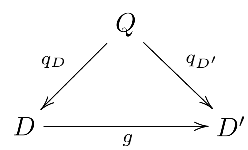
Definition 5.1.1 (Cone) *If* $\mathbf{D}$ *is a diagram in* $\mathcal{C}$*, a* cone over $\mathbf{D}$ *or a* $\mathbf{D}$-cone *consists of*

(i) *an object* Q *of* $\mathcal{C}$

(ii) *for every object* D *of* $\mathbf{D}$*, a unique arrow* $Q \stackrel{q_D}{\to} D$ *of* $\mathcal{C}$

such that

> *For any two objects* D *and* D' *of* $\mathbf{D}$ *and any arrow* $D \stackrel{g}{\to} D'$ *of* $\mathbf{D}$*, the following diagram commutes:*

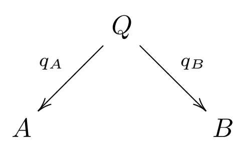


Example 4

- *Consider the following diagram:*

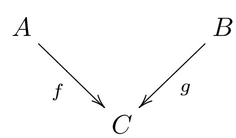
  $$A \qquad B$$

  A cone for this diagram would consist of an object Q *and arrows* $Q \stackrel{q_A}{\to} A$*, and* $Q \stackrel{q_B}{\to} B$ *such that the following diagram commutes:*

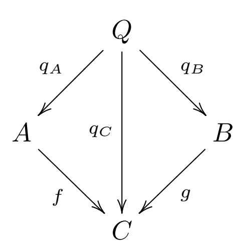


  Clearly any arrows with the appropriate domains and codomains will make the diagram commute since it contains no arrows.

- *Consider the following diagram:*

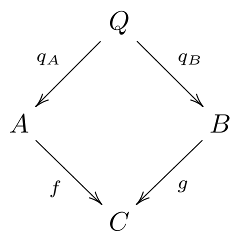


  A cone for this diagram would consist of an object Q *and arrows* $Q \stackrel{q_A}{\to} A$*,* $D \stackrel{q_B}{\to} B$*, and* $Q \stackrel{q_C}{\to} C$ *such that the following diagram commutes:*



In fact, in this case, we omit q_C *since it is entirely determined by the other arrows. It is determined by* q_A *and* f *or by* q_B *and* g*: since the diagram must commute, we have* $q_C = f \circ q_A = g \circ q_B$*.*

Thus, henceforth, we will say that a cone for this diagram consists of an object Q *and arrows* $Q \xrightarrow{q_A} A$ *and* $Q \xrightarrow{q_B} B$ *such that the following diagram commutes:*

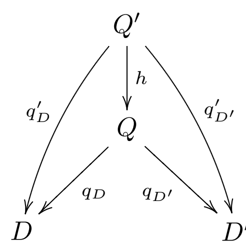


Now we come to define what it means for a cone over **D** to be a *limit over* **D**. Intuitively, it means that the cone is 'maximal' in the appropriate sense given by the arrows of the category.

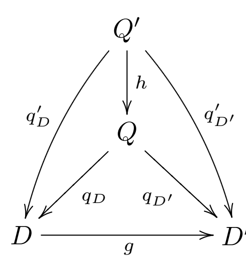
Definition 5.1.2 (Limit) *Suppose* **Q** *is a* **D***-cone. Then* **Q** *is a* limit over **D** *or a* **D**-limit *if*

For any **D***-cone* **Q′***, there is a unique arrow* $Q' \xrightarrow{h} Q$ *such that*

(1) *For any objects* D *and* D' *in* **D***, the following diagram commutes:*

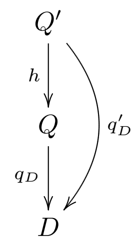


(2) *For any arrow* $D \xrightarrow{g} D'$ *in* **D***, the following diagram commutes:*



In fact, in this definition (2) follows from (1), and (1') follows from the following condition on **Q**:

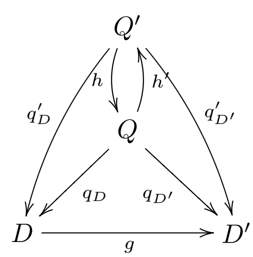
For any **D**-cone $\mathbf{Q'}$, there is a unique arrow $Q' \xrightarrow{h} Q$ such that, for any object D in **D**, the following diagram commutes:



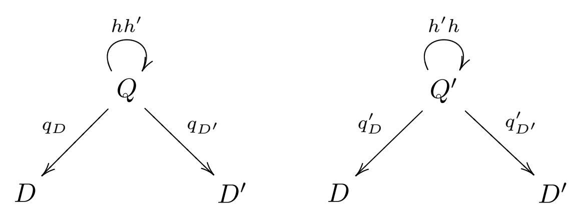
Proposition 5.1.3 *A limit over a diagram* **D** *is unique up to isomorphism.*

That is, if **Q** *and* $\mathbf{Q'}$ *are both limits over* **D**, *there is an iso* $Q \overset{f}{\cong} Q'$.

Proof. Since **Q** and $\mathbf{Q'}$ are both cones over **D**, there are unique arrows $Q \xrightarrow{h'} Q'$ and $Q' \xrightarrow{h} Q$ such that the following diagram commutes:



Thus, hh' and $h'h$ make the following diagrams commute:



But so do Id_Q and $\mathsf{Id}_{Q'}$ respectively. Thus, $\mathsf{Id}_Q = hh'$ and $\mathsf{Id}_{Q'} = h'h$. Thus, $Q \cong Q'$, as required. □

What's more,

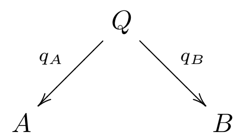
Proposition 5.1.4 *If* **Q** *is a limit over* **D** *and* $Q' \overset{h}{\cong} Q$, *then* $\mathbf{Q'}$ *is a limit over* **D**, *where* $q'_D = q_D \circ h$, *for all* D *in* **D**.

5.1.1 Definition of product

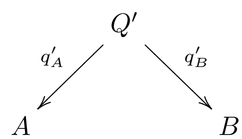
Example 5 *Consider again the following diagram:*

$$A \qquad B$$

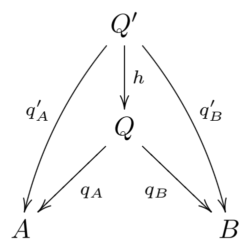
A limit for this diagram consists of a cone



with the following property: If



is a cone of this diagram, then there is a unique $Q' \xrightarrow{h} Q$ *such that the following diagram commutes:*



Limits of these diagrams are important in many areas of mathematics:

- In Set, the Cartesian product $A \times B$ of A and B, together with projection maps $p_A : (a, b) \mapsto a$ and $p_B : (a, b) \mapsto b$, is a limit for the diagram

$$A \qquad B$$

- In Grp, the direct products, together with projection maps, are limits for these diagrams.
- In a category based on a poset, the greatest lower bound of A and B (where it exists), together with the unique maps from this object to A and B respectively, is a limit of the diagram consisting only of A and B.

Definition 5.1.5 (Product) *We call a limit of the diagram*

$$A \qquad B$$

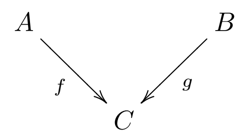
a product *of* A *and* B*, and write it* $A \times B$*.*

Notice that this definition says what it means to be *a* product of A and B. It does not attempt to pin down a unique object that we might call *the* product, whereas when we define the product of two sets as a set of Kuratowski ordered pairs, we do attempt to pin down such a unique object. It is exactly this sort of definition that is characteristic of category theory. Rather than explicitly defining a single object that does the job we want a product to do, the category theorist defines a *functional role* and says that *any* object that occupies that functional role is a product.

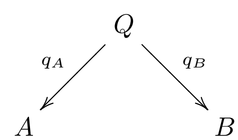
By Proposition 5.1.3, all products are isomorphic: that is, there is a left- and right-invertible map between them.

5.1.2 Definition of pullback and equalizer

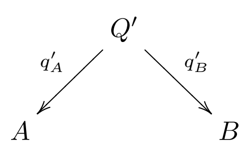
Example 6 *Consider again the following diagram:*



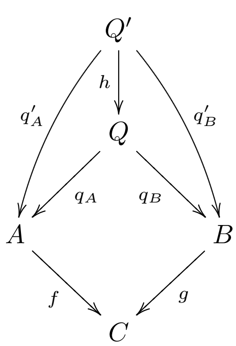
A limit for this diagram consists of a cone



with the following property: If

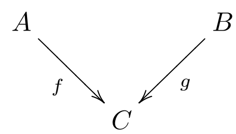


is a cone of this diagram, then there is a unique $Q' \xrightarrow{h} Q$ *such that the following diagram commutes:*



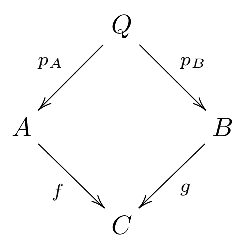
Limits of these diagrams will prove important in what follows. For the moment, we need only understand how they manifest themselves in Set. First, we give such limits a name:

Definition 5.1.6 (Pullback) *We call a limit of the diagram*



a pullback *for that diagram.*

We say that the following diagram in a pullback square *if* $A \overset{p_A}{\leftarrow} P \overset{p_B}{\rightarrow} B$ *is a pullback for* $A \overset{f}{\rightarrow} C \overset{g}{\leftarrow} B$*:*



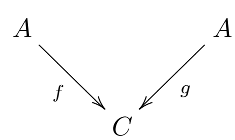
In Set, the following set together with the restrictions of the projection maps p_A and p_B is a pullback of $A \overset{f}{\rightarrow} C \overset{g}{\leftarrow} B$:

$$A \times_C B = \{(a, b) \in A \times B : f(a) = g(b)\}$$

Thus, if $A \overset{f,g}{\rightrightarrows} C$, then

$$\{x \in A : f(x) = g(x)\}$$

is a pullback of



A pullback of such a diagram is often called an *equalizer of* $A \overset{f,g}{\rightrightarrows} C$.

5.2 Cocones, Colimits, and Coproducts

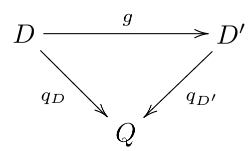
Definition 5.2.1 (Cocone) *If* **D** *is a diagram in* $\mathcal{C}$*, a* cocone over **D** *or a* **D**-cocone *consists of*

(i) *an object* Q *of* $\mathcal{C}$

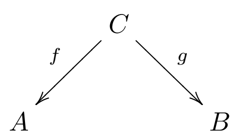
(ii) *for every object D of $\mathbf{D}$, a unique arrow $D \stackrel{q_D}{\to} Q$ of $\mathcal{C}$*

such that

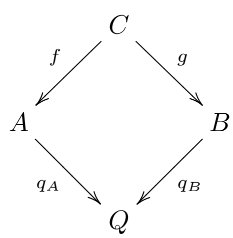
> *For any two objects D and D' of $\mathbf{D}$ and any arrow $D \stackrel{g}{\to} D'$ of $\mathbf{D}$, the following diagram commutes:*



Example 7 *Consider the following diagram:*



A cocone for this diagram would consist of an object Q and arrows $A \stackrel{q_A}{\to} Q$ and $B \stackrel{q_B}{\to} Q$ such that the following diagram commutes:



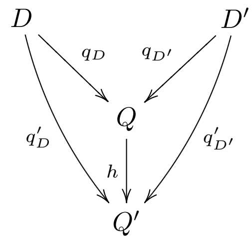
For the same reasons as above, we omit $C \stackrel{q_C}{\to} Q$ since it is entirely determined by the other arrows.

Now we define what it means for a cocone over $\mathbf{D}$ to be a *colimit over* $\mathbf{D}$. This time, it means that the cocone is 'minimal' in the appropriate sense given by the arrows.

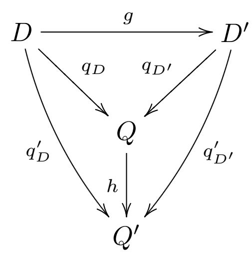
Definition 5.2.2 (Colimit) *Suppose* $\mathbf{Q}$ *is a* $\mathbf{D}$*-cocone. Then* $\mathbf{Q}$ *is a* colimit over $\mathbf{D}$ *or a* $\mathbf{D}$-colimit *if*

> *For any* $\mathbf{D}$*-cocone* $\mathbf{Q}'$*, there is a unique arrow $Q \stackrel{h}{\to} Q'$ such that*
>
> (1) *For any objects D and D' in $\mathbf{D}$, the following diagram com-*

mutes:

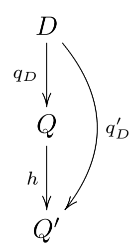


(2) *For any arrow* $D \xrightarrow{g} D'$ *in* $\mathbf{D}$*, the following diagram commutes:*



As in the definition of limit, (2) follows from (1), and (1) follows from the following condition on $\mathbf{Q}$:

> For any $\mathbf{D}$-cocone $\mathbf{Q'}$, there is a unique arrow $Q \xrightarrow{h} Q'$ such that, for any object D in $\mathbf{D}$, the following diagram commutes:



Proposition 5.2.3 *A colimit over a diagram* $\mathbf{D}$ *is unique up to isomorphism.*

That is, if $\mathbf{Q}$ *and* $\mathbf{Q'}$ *are both colimits over* $\mathbf{D}$*, there is an iso* $Q \overset{f}{\cong} Q'$.

Proof. This is exactly the co-proof of the proof of Proposition 5.1.3. □

What's more,

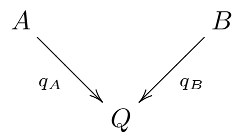
Proposition 5.2.4 *If* $\mathbf{Q}$ *is a colimit over* $\mathbf{D}$ *and* $Q \overset{h}{\cong} Q'$*, then* $\mathbf{Q'}$ *is a colimit over* $\mathbf{D}$*, where* $q'_D = h \circ q_D$*, for all* D *in* $\mathbf{D}$.

5.2.1 Definition of coproduct

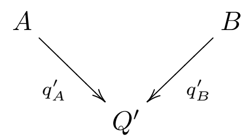
Example 8 *Consider again the diagram*

$$A \qquad B$$

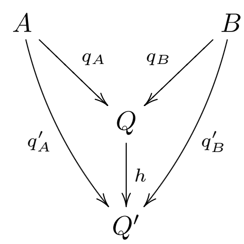
A colimit for this diagram consists of a cocone



such that if



is a cocone of this diagram, then there is a unique arrow $C \xrightarrow{h} C'$ *such that the following diagram commutes:*



Colimits of these diagrams are also important in many areas of mathematics:

- In Set, disjoint unions, together with injection maps, are colimits for these diagrams.

- In Grp, direct sums, together with injection maps, are colimits for these diagrams.

- In a category based on a poset, the least upper bound of A and B (where it exists), together with the unique maps from A and B to this object, is a colimit of the diagram consisting only of A and B.

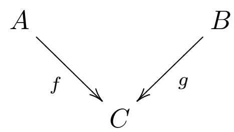
Definition 5.2.5 (Coproduct) *Given objects A and B, we call a colimit of the diagram*

$$A \qquad B$$

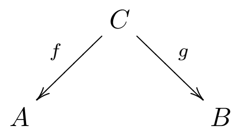
a coproduct of A and B, *and write it* $A + B$.

5.2.2 Definition of pushout and coequalizer

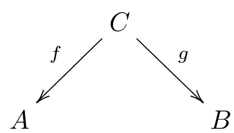
Just as we call the limit of the following diagram its pullback:



so we call the colimit of the following diagram its *pushout*:

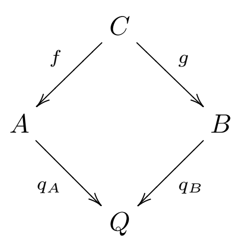


Definition 5.2.6 (Pushout) *We call a colimit of the diagram*



a pushout *for that diagram.*

We say that the following diagram in a pushout square *if* $A \stackrel{q_A}{\to} Q \stackrel{q_B}{\leftarrow} B$ *is a pushout for* $A \stackrel{f}{\leftarrow} C \stackrel{g}{\to} B$*:*



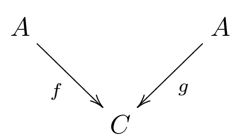
In Set, the following set together with the obvious inclusion maps i_A and i_B is a pushout of $B \stackrel{f}{\leftarrow} C \stackrel{g}{\to} A$:

$$A +_C B = \{(f(c), g(c)) : c \in C\} \cup (A - \mathrm{ran}(f)) \cup (B - \mathrm{ran}(g))$$

Thus, if $C \stackrel{f,g}{\rightrightarrows} A$, then

$$\{(f(c), g(c)) : c \in C\} \cup (A - \mathrm{ran}(f)) \cup (A - \mathrm{ran}(g))$$

is a pushout of



A pushout of such a diagram is often called an *coequalizer of* $C \overset{f,g}{\rightrightarrows} A$.

There is a better description of a *coequalizer* in Set, which we will employ in section 12.5 when we use coequalizers to give a particular definition of a *natural number object*. Given set functions $C \overset{f,g}{\rightrightarrows} A$, it is possible to construct a relation $R \subseteq A \times A$ on A such that

(i) R is an equivalence relation

(ii) $\{(f(x), g(x)) : x \in C\} \subseteq R$

(iii) R is the smallest subset of $A \times A$ that satisfies (i) and (ii).

Then the quotient set B/R of equivalence classes of R in B, together with the natural function $B \overset{e}{\to} B/R$ that takes an element of B to the equivalence class it inhabits, is a coequalizer of $C \overset{f,g}{\rightrightarrows} A$.

5.3 The second axiom of a topos

We have seen that taking limits and colimits of finite diagrams to exist gives us analogues of the set-theoretical notions of product, disjoint union, union, and intersection. This leads to the second axiom of a topos:

Axiom 2 *A topos contains limits and colimits for all finite diagrams.*

Thus, a topos contains products, coproducts, pullbacks, and pushouts for all relevant diagrams.

Chapter 6

Initial and terminal objects

Consider the *empty diagram* in the category $\mathcal{C}$. That is, the diagram that contains no objects and no arrows. Then its cones and cocones are just the objects of $\mathcal{C}$. Thus:

- If the empty diagram has a limit, it is a object 1 such that, for every $\mathcal{C}$-object A, there is a unique arrow $A \stackrel{1_A}{\to} 1$.
- If the empty diagram has a colimit, it is a $\mathcal{C}$-object 0 such that, for every $\mathcal{C}$-object A, there is a unique arrow $0 \stackrel{0_A}{\to} A$.

Definition 6.0.1 (Initial object) *A limit of the empty diagram (if it exists) is called a* terminal object *of* $\mathcal{C}$*.*

Definition 6.0.2 (Terminal object) *A colimit of the empty diagram (if it exists) is called an* initial object *of* $\mathcal{C}$*.*

By Propositions 5.1.3 and 5.2.3, a terminal or initial object is unique up to isomorphism. By Axiom 2, every topos has initial and terminal objects.

Example 9

- *In* Set*,* $\varnothing$ *is the only initial object and any singleton* $\{a\}$ *is a terminal object.*
- *In* Grp*, the trivial group* $\{e_G\}$ *is both initial and terminal. Such objects are called* zero objects*.*
- *In a category based on a poset, any minimum element is an initial object (if such an element exists) and any maximum element is a terminal object (if such an element exists).*
- *If* $\mathcal{C}$ *is a category, then* $X \stackrel{\mathsf{Id}_X}{\to} X$ *is a terminal object of* $\mathcal{C}/X$*.*

  If $\mathcal{C}$ *has an initial object* 0*, then* $0 \stackrel{0_X}{\to} X$ *is an initial object of* $\mathcal{C}/X$*.*

Proposition 6.0.3 *If* 1 *is a terminal object and* $1 \xrightarrow{f} A$*, then* f *is monic.*

Proof. If $B \overset{g,h}{\rightrightarrows} 1$, then $g = 1_B = h$, by definition of a terminal object. □

Proposition 6.0.4 *If* 1 *is a terminal object, then* $1 \times A \cong A \cong A \times 1$.

Chapter 7

Members of objects

The fundamental notion in the traditional theory of sets is the membership relation. As I mentioned above, it plays a minor role in the categorial theory of sets. But it is essential nonetheless. How are we to understand this relation in the category Set or, more generally, in a topos? The answer is given by Lawvere [**?**].

As is common when devising categorial analogues for non-categorial definitions, we consider the action associated with the defined object. The action associated with the notion of set membership is that of *picking out* the member in question. In Set, the action of picking out a member of a set A can be understood as a function from a singleton set into A: the value of the function at the single element of the singleton is the member. Recall that any singleton is a terminal object and any terminal object is a singleton in Set. Thus, we have the following definition:

Definition 7.0.5 (Member of) *If $\mathcal{C}$ is a category with a terminal object 1 and A is an object of $\mathcal{C}$, then a* member of A *is an arrow $1 \xrightarrow{x} A$.*

It is immediately clear from this definition that there can be no 'membership chains' on this definition. After all, it does not make sense to ask of a member $1 \xrightarrow{x} A$ of A whether some other arrow y is a member of x. Members are *members of objects* not *members of arrows*; and a member is itself an arrow; thus, there can be no members of members.

We will rarely use the notion of member in our definitions, but it will prove useful as a way of checking that our categorial definitions are genuinely analogues to their non-categorial counterparts. However, we can say something:

Proposition 7.0.6

(1) 1 *has exactly one element.*

(2) *If 0 has an element, then $0 \cong 1$.*

And we can also say what it means for an arrow to be *injective* and *surjective*.

Definition 7.0.7 (Injective and surjective) *Given* $A \xrightarrow{f} B$*, we say that*

(1) f is injective *if, for all* $1 \overset{x,y}{\rightrightarrows} A$*, iff* $fx = fy$*, then* $x = y$*.*

(2) f is surjective *if, for all* $1 \xrightarrow{y} B$*, there is* $1 \xrightarrow{x} A$ *such that* $fx = y$*.*

As we know, in Set the injective arrows are just the monics, and the surjective arrows are just the epics. But this need not be true in all categories with terminal objects. However, as we will see in section 12.2, it is true in all categories that share an important feature with Set.

Chapter 8

Exponential objects

Axiom 2 ensures that all toposes have analogues of the addition and multiplication operations on sets—they are coproducts and products, respectively.

However, it does not guarantee that they all have analogues of power sets $P(A)$, nor *a fortiori* that they all have analogues of function spaces or exponential sets $B^A = \{f : A \to B\}$ (where here f ranges over set-theoretical functions from A to B; that is, a certain subset of the Cartesian product of A and B). In this section, we introduce an axiom that does guarantee these objects.

Forget category theory for the moment, and think about sets. Consider a (set-theoretical) function $f : A \times C \to B$. Then, for every element $c \in C$, $f_c : a \mapsto f(a, c)$ is a function from A to B. Then $\hat{f} : c \mapsto f_c$ is a function from C into B^A. Moreover, there is a function $ev : A \times B^A \to B$ such that $ev : (a, f) \mapsto f(a)$. We call this the evaluation function of B^A, for it evaluates a function from A to B at a value in A. Thus,

$$ev(a, f_c) = f_c(a) = f(a, c)$$

Returning to category theory, we use this insight to define an exponential object.

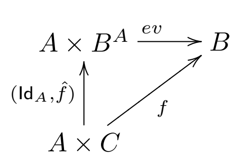
Definition 8.0.8 (Exponential) *Suppose $\mathcal{C}$ is a category with products. For any objects A and B, an* exponential of A and B *consists of*

(i) *an object B^A of category $\mathcal{C}$*

(ii) *an arrow $A \times B^A \xrightarrow{ev} B$ of category $\mathcal{C}$*

such that

> *For any arrow $A \times C \xrightarrow{f} B$, there is an arrow $C \xrightarrow{\hat{f}} B^A$ such that the following diagram commutes:*



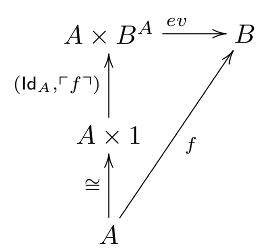
Proposition 8.0.9 *Exponentials of A and B are unique up to isomorphism and anything isomorphic to an exponential of A and B is itself an exponential of A and B.*

This is a good example of a case in which we can check that we are on the right track by considering whether the exponential B^A satisfies certain basic desiderata stated in terms of the membership relation. In set theory, we wish an exponential object to have the following property:

f is a member of B^A iff $f : A \to B$

We do not get exactly this result, since, if $A \xrightarrow{f} B$, then f is the wrong sort of thing to be a member of B^A (its domain is not 1 and its codomain is not B^A). But we do get a close analogue. We can show that there is a one-one correspondence between arrows $A \xrightarrow{f} B$ and arrows $1 \xrightarrow{g} B^A$.

Definition 8.0.10 (Name of an arrow) *If $A \xrightarrow{f} B$ is an arrow, the* name of f *(written $\ulcorner f \urcorner$) is the arrow $1 \xrightarrow{\ulcorner f \urcorner} B^A$ such that the following diagram commutes:*



Proposition 8.0.11 *There is a bijection $f \mapsto \ulcorner f \urcorner$ between the set of arrows $A \xrightarrow{f} B$ and the set of arrows $1 \xrightarrow{\ulcorner f \urcorner} B^A$.*

With the definition of an exponential in hand, we can introduce the third axiom of a topos.

Axiom 3 *A topos has exponentials for every pair of objects.*

Definition 8.0.12 (Cartesian closed category) *A category with limits for all finite diagrams and exponentials for all pairs of objects is called a* Cartesian closed category*.*

Chapter 9

Subobjects and their classifiers

9.1 Subobjects

We saw above that, in the categorial theory of sets, a member of a set A is an *arrow*, not an *object*. In particular, it is an arrow from a terminal object 1 into A. Thus, a member of a set A is a function from a singleton into A, not another set B.

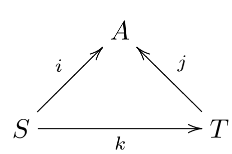
Similarly, the categorial analogy of a *subset* of a set A—which we call a *subobject of* A—is an arrow, not an object. In particular, it is a monic arrow from an object S into A. The idea is that, just as an arrow $1 \xrightarrow{x} A$ 'picks out' a single element of A, a monic arrow $S \overset{i}{\rightarrowtail} A$ 'picks out' a subset of A, namely, its range.

Definition 9.1.1 (Part or subobject) *Suppose A is an object. Then a* subobject of A *is a monic arrow $S \overset{i}{\rightarrowtail} A$.*

By Proposition 6.0.3 above, any member of A is a subobject of A. Thus, in the categorial treatment of sets, members are simply subobjects with the terminal object 1 as the domain. Thus, there is no distinction between a one-element subset of a set and a member of that set.

Definition 9.1.2 (Inclusion) *Suppose $S \overset{i}{\rightarrowtail} A$ and $T \overset{j}{\rightarrowtail} A$ are subobjects of A. Then we say that S* is included in T *(written $S \subseteq T$) if*

> *There is $S \xrightarrow{k} T$ such that the following diagram commutes:*



9.2 Subobject classifiers

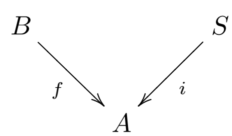
9.2.1 Motivation

Our next step is to introduce into a topos an object Ω whose members will act as the 'truth-values' for our topos, and then to associate with each subobject $S \stackrel{i}{\rightarrowtail} A$ a 'characteristic function' $A \stackrel{\chi_i}{\to} \Omega$.

Again, forget category theory and think about sets. Thus, $\Omega = \{\mathsf{true}, \mathsf{false}\}$. One way to think of the characteristic function of a subset $S \subseteq A$ is as the function χ_S from A to $\{\mathsf{true}, \mathsf{false}\}$ such that the inverse image of the subset $\{\mathsf{true}\} \subseteq \{\mathsf{true}, \mathsf{false}\}$ under χ_S is S. Thus, we will be able to give our definition of a characteristic function if we can give a definition of the inverse image of a subobject $S \stackrel{i}{\rightarrowtail} A$ of the object A under a function $B \stackrel{f}{\to} A$ into A.

9.2.2 Inverse images of subobjects

Recall that, in Set, one particular pullback for the following diagram:



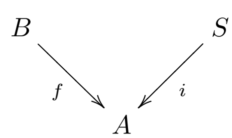
is the following set:

$$\{(b, s) : f(b) = i(s)\}$$

Now suppose that $S \stackrel{i}{\rightarrowtail} A$ and i is the inclusion map $i : s \mapsto s$, for every s in S. Then one particular pullback is the set

$$\{(b, s) \in B \times S : f(b) = i(s) = s\} \cong \{b \in B : f(b) \in S\} =_{df.} f^{-1}(S)$$

Definition 9.2.1 (Inverse image) *If $S \stackrel{i}{\rightarrowtail} A$ and $B \stackrel{f}{\to} A$, then the pullback of the diagram*



is called an inverse image of $S \stackrel{i}{\rightarrowtail} A$ under f *(written $f^{-1}(S)$).*

9.2.3 The definition

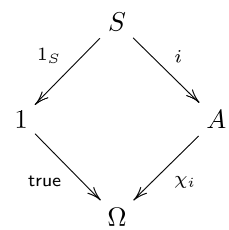
Thus, we will define the characteristic function χ_i of a subobject $S \stackrel{i}{\rightarrowtail} A$ to be the unique function such that S is an inverse image of $1 \stackrel{\mathsf{true}}{\rightarrowtail} \Omega$ under χ_i.

Definition 9.2.2 (Subobject classifier) *Suppose $\mathcal{C}$ is a category with a terminal object* 1. *Then a* subobject classifier in $\mathcal{C}$ *consists of*

(i) *an object* Ω *of category* $\mathcal{C}$

(ii) *an arrow* $1 \xrightarrow{\mathsf{true}} \Omega$

such that

> *For any object* A *and subobject* $S \overset{i}{\rightarrowtail} A$*, there is a unique arrow* $A \xrightarrow{\chi_i} \Omega$ *such that:*
>
> 
>
>
> *is a pullback square.*

Proposition 9.2.3 *Subobject classifiers are unique up to isomorphism and anything isomorphic to a subobject classifier is itself a subobject classifier.*

We can now state the final axiom of toposes:

Axiom 4 *A topos has a subobject classifier.*

The definition of the truth-value true is built into our definition of a subobject classifier. But how are we to define the truth-value false? Does every subobject classifier have such a truth-value? Given our understanding of characteristic functions, it makes sense to define false as the member of Ω that is the characteristic function of the 'empty' subobject $0 \overset{0_1}{\rightarrowtail} 1$. Thus,

Definition 9.2.4 (false) $1 \xrightarrow{\mathsf{false}} \Omega$ *is the characteristic function of the subobject* $0 \overset{0_1}{\rightarrowtail} 1$.

Chapter 10

The definition of a topos

10.1 The definition

We have now stated all the axioms for a topos. Thus, we state the definition:

Definition 10.1.1 (Topos) *A* topos *is a category with*

(i) *Limits and colimits for all finite diagrams*

(ii) *Exponentials for all pairs of objects*

(iii) *A subobject classifier.*

In fact, this definition can be greatly simplified by the following theorem, which provides an invaluable tool when we come to show that a particular category is a topos:

Theorem 10.1.2 *If $\mathcal{E}$ is a category with a terminal object, pullbacks, exponentials, and a subobject classifier, then $\mathcal{E}$ is a topos.*

10.2 Some examples

The most obvious example and indeed the example that motivated much of what was said above:

Example 10 Set *is a topos.*

- *Terminal objects in* Set*: singletons* $\{a\}$
- *Pullback of* $A \xrightarrow{f} C \xleftarrow{g} B$ *in* Set*:* $A \xleftarrow{p_A} A \times_C B \xrightarrow{p_B} B$
- *Exponentials in* Set*:* $B^A = \{f : A \to B\}$ *plus natural evaluation function.*
- *Subobject classifier in* Set*:* $\{a\} \xrightarrow{\mathsf{true}} \{\mathsf{true}, \mathsf{false}\}$ *such that* $\mathsf{true}(a) = \mathsf{true}$.

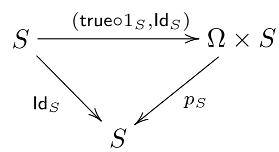
Using Theorem 10.1.2, we can prove what Freyd calls the *Fundamental Theorem of Toposes*:

Theorem 10.2.1 (Fundamental Theorem of Toposes) *If $\mathcal{E}$ is a topos and X is an object in $\mathcal{E}$, then $\mathcal{E}/X$ is a topos.*

Thus,

Example 11 *For any set S, Set/S is a topos.*

- *Terminal object in* Set/S*:* $S \overset{\mathsf{Id}_S}{\to} S$.
- *Subobject classifier in* Set*:*



Also using Theorem 10.1.2, we can give two further examples:

Example 12 *If $\mathcal{E}$ is a topos, then $\mathcal{E}^2$ is a topos.*

- *Terminal object in* $\mathcal{E}^2$*:* $(1, 1)$
- *Pullback of* $(A_1, A_2) \overset{(f_1,f_2)}{\to} (C_1, C_2) \overset{(g_1,g_2)}{\leftarrow} (B_1, B_2)$ *in* $\mathcal{E}^2$*:*

$$(A_1, A_2) \overset{(p_{A_1},p_{A_2})}{\leftarrow} (P_1, P_2) \overset{(p_{B_1},p_{B_2})}{\to} (B_1, B_2)$$

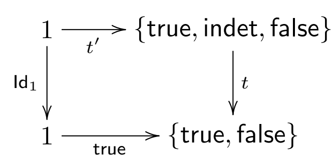
  where

  - $A_1 \overset{p_{A_1}}{\leftarrow} P_1 \overset{p_{B_1}}{\to} B_1$ *is a pullback of* $A_1 \overset{f_1}{\to} C_1 \overset{g_1}{\leftarrow} B_1$ *and*
  - $A_2 \overset{p_{A_2}}{\leftarrow} P_2 \overset{p_{B_2}}{\to} B_2$ *is a pullback of* $A_2 \overset{f_2}{\to} C_2 \overset{g_2}{\leftarrow} B_2$

- *Exponential in* $\mathcal{E}^2$*:* $(B_1, B_2)^{(A_1,A_2)} = (B_1^{A_1}, B_2^{A_2})$ *with* $ev_{\mathcal{E}^2} = (ev_{\mathcal{E}}, ev_{\mathcal{E}})$.
- *Subobject classifier in* $\mathcal{E}^2$*:* $(1, 1) \overset{(\mathsf{true},\mathsf{true})}{\to} (\Omega, \Omega)$.

Example 13 $\mathsf{Set}^{\to}$ *is a topos.*

- *Terminal object in* $\mathsf{Set}^{\to}$*:* $1 \overset{\mathsf{Id}_1}{\to} 1$
- *Subobject classifier in* $\mathsf{Set}^{\to}$*:*



  where

- $t'(a) = \mathsf{true}$ *and* $\mathsf{true}(a) = \mathsf{true}$ (*as usual*)
- $t(\mathsf{false}) = \mathsf{false}$ *and* $t(\mathsf{indet}) = t(\mathsf{true}) = \mathsf{true}$

The idea is that, if

$$(S_1 \xrightarrow{\xi_S} S_2) \overset{(i_1,i_2)}{\rightarrowtail} (A_1 \xrightarrow{\xi_A} A_2)$$

is a subobject of $A_1 \xrightarrow{\xi_A} A_2$*, then its characteristic function is* (χ_1, χ_2) *defined as follows:*

$$\chi_1(x) = \begin{cases} \mathsf{true} & \textit{if } x \in S_1 \ \& \ \xi_S(x) \in S_2 \\ \mathsf{indet} & \textit{if } x \notin S_1 \ \& \ \xi_S(x) \in S_2 \\ \mathsf{false} & \textit{if } x \notin S_1 \ \& \ \xi_S(x) \notin S_2 \end{cases}$$

and

$$\chi_2(y) = \begin{cases} \mathsf{true} & \textit{if } y \in S_2 \\ \mathsf{false} & \textit{if } y \notin S_2 \end{cases}$$

Chapter 11

The algebra of subobjects

In set theory, it is well-known that the set of subsets of a given set A is a Boolean algebra under the inclusion relation $\subseteq$.

Definition 11.0.2 (Boolean algebra) *A partially ordered set* $(A, \leq)$ *is a* Boolean algebra *if*

(1) $(A, \leq)$ *is a* lattice. *That is, for any* $x, y \in A$*, there is*

  (i) *a greatest lower bound of* x *and* y*, which we write* $x \wedge y$

  (ii) *a least upper bound of* x *and* y*, which we write* $x \vee y$

(1) $(A, \leq)$ *has* extreme elements. *That is,*

  (i) *there is a minimum element* $0 \in A$

  (ii) *there is a maximum element* $1 \in A$

(3) $(A, \leq)$ *is* distributive. *That is, for any* $x, y, z \in A$*,*

  (i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

  (ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(4) $(A, \leq)$ *is* complemented. *That is, for any* $x \in A$*, there is* $\neg x \in A$ *such that*

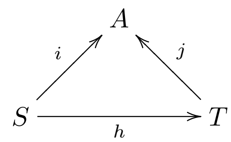
  (i) $x \wedge \neg x = 0$

  (ii) $x \vee \neg x = 1$

It turns out that all but (4) holds true in any topos.

Definition 11.0.3 ($\mathrm{Sub}(A)$**)** *Given an object* A *of a topos* $\mathcal{E}$*, let* $\mathrm{Sub}(A)$ *be the set of equivalence classes of subobjects of* A *in* $\mathcal{E}$ *under the following equivalence relation:*

$S \cong T$ iff there is an iso $S \overset{h}{\cong} T$ such that the following diagram commutes:



We will abuse notation and talk of a subobject $S \overset{i}{\rightarrowtail} A$ as a member of $\mathrm{Sub}(A)$ when we really mean that the equivalence class to which it belongs is a member of that set.

Immediately, we can see:

Proposition 11.0.4 *In* $(\mathrm{Sub}(A), \subseteq)$,

(1) $0 \overset{0_A}{\rightarrowtail} A$ *is a minimum element*

(2) $A \overset{\mathsf{Id}_A}{\rightarrowtail} A$ *is a maximum element.*

Next, we define the greatest lower bounds and least upper bounds for pairs of subobjects $S \overset{i}{\rightarrowtail} A$ and $T \overset{j}{\rightarrowtail} A$ in $(\mathrm{Sub}(A), \subseteq)$.

First, the greatest lower bound. Forget category theory and think about sets. Recall that one pullback of $S \overset{i}{\to} A \overset{j}{\leftarrow} T$ is

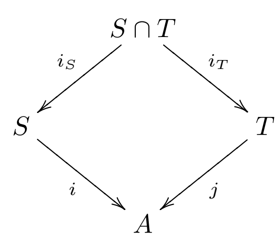
$$S \times_A T = \{(s,t) \in S \times T : i(s) = j(t)\}$$

But, since i and j are inclusions,

$$S \times_A T = \{(s,t) \in S \times T : i(s) = j(t)\} = \{(s,t) \in S \times T : s = t\} \cong S \cap T$$

Now we return to category theory:

Definition 11.0.5 (Intersection) *Suppose $S \overset{i}{\to} A$ and $T \overset{j}{\to} A$. Then an* intersection of S and T *is $S \overset{i_S}{\leftarrow} S \cap T \overset{i_T}{\to} T$ such that the followig diagram is a pullback square:*



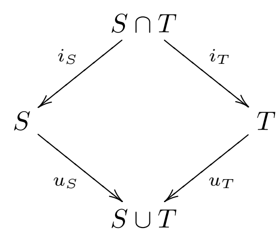
It is easy to see that $S \cap T \overset{i \circ i_S = j \circ i_T}{\to} A$ is a monic, and thus a subobject of A.

Next, the least upper bound. Again, forget category theory and think about sets. Recall that one pushout of $S \overset{i_S}{\leftarrow} S \cap T \overset{i_T}{\rightarrow} T$ is

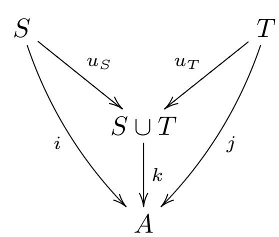
$$\begin{aligned} S +_{S\cap T} T &= \{(i_S(x), i_T(x)) : x \in S \cap T\} \cup (S - \mathrm{ran}(i_S)) \cup (T - \mathrm{ran}(i_T)) \\ &\cong S \cap T \cup (S - S \cap T) \cup (T - S \cap T) \\ &= S \cup T \end{aligned}$$

Now we return to category theory:

Definition 11.0.6 (Union) *Suppose $S \overset{i}{\rightarrow} A$ and $T \overset{j}{\rightarrow} A$. Then a* union of S and T *is $S \overset{u_S}{\rightarrow} S \cup T \overset{u_T}{\leftarrow} T$ such that the following diagram is a pushout square:*



It is easy to see that $S \cup T \overset{k}{\rightarrow} A$ is monic, and thus a subobject of A, where k is the unique arrow that makes the following diagram commute:



The following proposition follows easily from the fact that $S \cap T$ is a *limit* and $S \cup T$ is a *colimit*:

Proposition 11.0.7 *In* $(\mathrm{Sub}(A), \subseteq)$,

(1) *$S \cap T$ is the greatest lower bound of S and T*

(2) *$S \cup T$ is the least upper bound of S and T.*

And these operations satisfy de Morgan's laws of distributivity.

Thus, combining Propositions 11.0.4 and 11.0.7, we have:

Proposition 11.0.8 *For any topos $\mathcal{E}$ and any object A of $\mathcal{E}$,* $(\mathrm{Sub}(A), \subseteq)$ *is a distributive lattice with extreme elements.*

We cannot strengthen this result to show that $(\mathrm{Sub}(A), \subseteq)$ is always a Boolean algebra. For instance,

Proposition 11.0.9 *In* $\mathsf{Set}^{\rightarrow}$, $(\mathrm{Sub}(A), \subseteq)$ *is not necessarily complemented.*

Proof. In particular,

$$1_{\mathsf{Set}^{\rightarrow}} \overset{\mathsf{true}_{\mathsf{Set}^{\rightarrow}}}{\longrightarrow} \Omega_{\mathsf{Set}^{\rightarrow}}$$

does not have a complement in $(\mathrm{Sub}(\Omega_{\mathsf{Set}^{\rightarrow}}), \subseteq)$. □

Chapter 12

Kinds of topos

In this section, we survey some different types of topos and the relationships that hold between them.

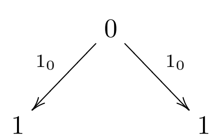
12.1 Non-degenerate toposes

A notable feature of Set is that the initial and terminal objects are distinct: that is, there is no initial element that is also a terminal element. As a category, this distinguishes it from Grp in which any trivial group is both an initial and terminal element. It is possible to prove that the only topos that has an initial element that is also a terminal element is the degenerate topos with one object and one arrow. Thus,

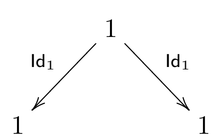
Definition 12.1.1 (Non-degenerate) *A topos is* non-degenerate *if* $0 \not\cong 1$.

Proposition 12.1.2 *In a topos* $\mathcal{E}$*, if* $\mathsf{true} = \mathsf{false}$*, then* $\mathcal{E}$ *is degenerate.*

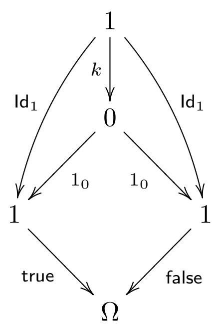
Proof. By the definition of false, the following diagram



is a pullback of $1 \overset{\mathsf{true}}{\to} \Omega \overset{\mathsf{false}}{\leftarrow} 1$. But, if $\mathsf{true} = \mathsf{false}$, then the following diagram is also a cone of this diagram:



Thus, there is a unique arrow $1 \xrightarrow{k} 0$ such that the following diagram commutes:



Then $0 \xrightarrow{k \circ 1_0} 0$ and $1 \xrightarrow{1_0 \circ k} 1$. But since 0 is an initial object and 1 is a terminal object, there is exactly one arrow from 0 to itself—namely, Id_0—and exactly one arrow from 1 to itself—namely, Id_1. Thus, $k \circ 1_0 = \mathsf{Id}_0$ and $1_0 \circ k = \mathsf{Id}_1$. Thus, $0 \xrightarrow{k} 1$, as required. □

12.2 Well-pointed toposes

One characteristic feature of Set is that it satisfies the Axiom of Extensionality. How might we state this condition using the language of objects and arrows? In set theory, we state it by saying that any distinct sets differ by at least one element. However, given our definition of membership in category theory as an arrow $1 \xrightarrow{x} A$, it is impossible that two objects share an element in common. Thus, the Axiom of Extensionality holds trivially for any category with a terminal object. Thus, we must try harder. The Axiom of Extensionality that we give is in fact an Axiom of Extensionality *for Functions*: that is, it says that two distinct functions from A to B differ in the value they assign to at least member of A. To state this axiom, we need the following definition:

Definition 12.2.1 (Separator) *If $\mathcal{C}$ is a category, a* separator for $\mathcal{C}$ *consists of*

(i) *an object S of category $\mathcal{C}$*

such that

> *For any arrows $A \overset{f,g}{\rightrightarrows} A$, if $f \neq g$, then there is $S \xrightarrow{x} A$ such that $fx \neq gx$.*

Thus, our axiom of extensionality for functions says that 1 is a separator. A topos that satisfies this axiom is called *well-pointed*.

Definition 12.2.2 (Well-pointed topos) *A topos is* well-pointed *if its terminal object is a separator.*

Proposition 12.2.3 *If $\mathcal{E}$ is well-pointed, then*

(1) *f is injective iff f is monic*

(2) *f is surjective iff f is epic*

12.3 Bivalent toposes

Definition 12.3.1 (Bivalent) *A topos $\mathcal{E}$ is* bivalent *if the only two truth-values in its subobject classifier are* true *and* false.

Proposition 12.3.2 *If $\mathcal{E}$ is well-pointed, then $\mathcal{E}$ is bivalent.*

Example 14

- Set *is bivalent.*
- Set^2 *is not bivalent.*
- $\mathsf{Set}^{\rightarrow}$ *is not bivalent.*

12.4 Boolean toposes

Definition 12.4.1 (Boolean) *A topos $\mathcal{E}$ is* Boolean *if, for every object A of $\mathcal{E}$, $(\mathrm{Sub}(A), \subseteq)$ is a Boolean algebra.*

There are various equivalent versions of this definition:

Proposition 12.4.2 *If $\mathcal{E}$ is a topos, the following statements are equivalent:*

(1) *$\mathcal{E}$ is Boolean.*

(2) *$(\mathrm{Sub}(\Omega), \subseteq)$ is a Boolean algebra.*

(3) *$1 \overset{\mathsf{false}}{\rightarrow} \Omega$ is the complement of $1 \overset{\mathsf{true}}{\rightarrow} \Omega$*

Proposition 12.4.3 *If $\mathcal{E}$ is well-pointed, then $\mathcal{E}$ is Boolean.*

Example 15

- Set *is Boolean.*
- Set^2 *is Boolean.*
- $\mathsf{Set}^{\rightarrow}$ *is not Boolean.*

Interestingly, there are also toposes that are bivalent, but not Boolean.

12.5 Natural number objects

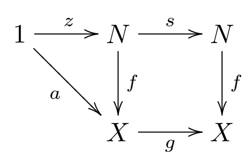
A topos need not contain any infinite objects. But we may wish to restrict attention to those that do. We do this first by defining the notion of a *natural number object*, due to Lawvere [**?**], though we will then state equivalent definitions due to Dedekind and Freyd.

On Lawvere's definition, a natural number object in a category $\mathcal{C}$ with a terminal element is an object N of $\mathcal{C}$ equipped with a distinguished member $1 \xrightarrow{z} N$ and a successor function $N \xrightarrow{s} N$ such that, for any object X equipped with a distinguished member $1 \xrightarrow{a} X$ and a successor function $X \xrightarrow{g} X$, there is a unique function $N \xrightarrow{f} X$ that satisfies the recursion equations:

$$\begin{aligned} fz &= a \\ fs &= gf \end{aligned}$$

Definition 12.5.1 (Natural number object - Lawvere) *Given an object* N *and arrows* $1 \xrightarrow{z} N \xrightarrow{s} N$*, we say that* $1 \xrightarrow{z} N \xrightarrow{s} N$ satisfies the Lawvere axiom for a natural number object *if*

> *For any object* X *with arrows* $1 \xrightarrow{a} X \xrightarrow{g} X$*, there is a unique arrow* $N \xrightarrow{f} X$ *such that the following diagram commutes:*



Lawvere's definition of a natural number object in a topos provides an excellent example of the difference between category-theoretical definitions and the more conventional set-theoretic definitions of mathematics.

On Dedekind's more conventional definition, a natural number object (in his terminology, a *simply infinite system*) is a *chain* S *of* z *under* s whenever S is a set, $z \in S$, and $s : S \to S$ is an injection with $sx \neq z$ for all $x \in S$: that is, S is the smallest subset of S that contains z and is closed under s. That is, Dedekind's definition specifies what a natural number object *is*. And, in his recursion theorem, he derives what it *does*.

Characteristically, Lawvere's category-theoretical definition turns things around. It specifies what a natural number object *does*—in particular, it satisfies Dedekind's recursion theorem. And we can use this fact to derive Dedekind's characterization of what it *is* (if we wish):

Definition 12.5.2 (Natural number object - Dedekind) *Given an object* N *and arrows* $1 \xrightarrow{z} N \xrightarrow{s} N$*, we say that* $1 \xrightarrow{z} N \xrightarrow{s} N$ satisfies the Dedekind axioms for a natural number object *if*

(1) *If* $1 \xrightarrow{x} N$*, then* $sx \neq z$

(2) *s is monic.*

(3) *If $S \overset{i}{\rightarrowtail} N$ is a subobject of N such that*

(a) *There is $1 \xrightarrow{z'} S$ such that $iz' = z$ and*

(b) *There is $S \xrightarrow{s'} S$ such that $is' = si$*

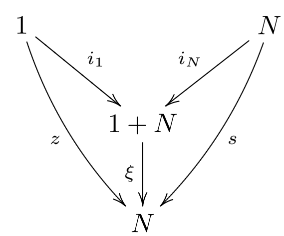
Then $S \overset{i}{\cong} N$.

Notice that (2) is a slightly stronger requirement than Dedekind imposed. He demanded only that s be an injection. While all monics are injections in Set, this is not necessarily the case in other toposes.

As we will see below, Lawvere's definition of a natural number object is equivalent to Dedekind's definition. Moreover, there is a third equivalent characterization of natural number objects, which is due to Freyd [**?**]:

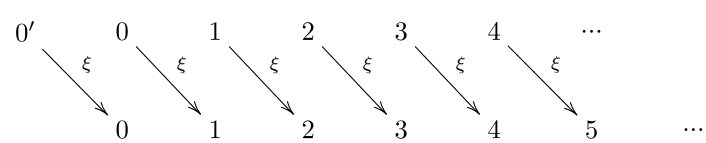
Definition 12.5.3 (Natural number object - Freyd) *Given an object N and arrows $1 \xrightarrow{z} N \xrightarrow{s} N$, we say that $1 \xrightarrow{z} N \xrightarrow{s} N$* satisfies the Freyd axioms for a natural number object *if*

(i) *If $1 \xrightarrow{i_1} 1 + N \xleftarrow{i_N} N$ is the coproduct of 1 and N, then the unique arrow $1 + N \xrightarrow{\xi} N$ that makes the following diagram commutes is an iso:*



(ii) *$N \xrightarrow{1_N} 1$ is a coequalizer for $N \overset{s, \mathsf{Id}_N}{\rightrightarrows} N$.*

Intuitively, Freyd's first axiom says that the following function is an isomorphism:



His second axiom essentially amounts to Dedekind's *chain condition* or Peano's axiom of induction: that is, it says that N is the smallest object to contain z and be closed under successor. To see this, consider coequalizers in the category Set.

Forget category theory and think about sets for a moment. Now, consider the arrows (functions) $A \overset{f,g}{\rightrightarrows} B$. Then there is $R \subseteq B \times B$ such that

(i) R is an equivalence relation

(ii) $\{(f(x), g(x)) : x \in A\} \subseteq R$

(iii) R is the smallest subset of $B \times B$ that satisfies (i) and (ii).

Then the quotient set B/R of equivalence classes of R in B, together with the natural function $B \xrightarrow{e} B/R$ that takes an element of B to the equivalence class it inhabits, is a coequalizer of $A \overset{f,g}{\rightrightarrows} B$.

Now consider the arrows $N \overset{s,\mathsf{Id}_N}{\rightrightarrows} N$. Then a coequalizer of these arrows is the set of equivalence classes of the smallest equivalence relation R that contains $\{(n, s(n)) : n \in N\}$. Thus, in Set, Freyd's second axiom says that there is just one equivalence class of N under R, which thus contains all of N. This amounts to saying that N is the smallest set that contains z and is closed under s.

Theorem 12.5.4 *Given an object N and arrows $1 \xrightarrow{z} N \xrightarrow{s} N$, the following are equivalent in a non-degenerate topos:*

(1) $1 \xrightarrow{z} N \xrightarrow{s} N$ *satisfies the Lawvere axioms for a natural number object.*

(2) $1 \xrightarrow{z} N \xrightarrow{s} N$ *satisfies the Dedekind axioms for a natural number object.*

(3) $1 \xrightarrow{z} N \xrightarrow{s} N$ *satisfies the Freyd axioms for a natural number object.*

Proof. We prove (1) $\Rightarrow$ (2) below. □

Definition 12.5.5 (Natural number object) *If $\mathcal{C}$ is a category with a terminal object* 1*, a* natural number object of $\mathcal{C}$ *consists of*

(i) *an object N*

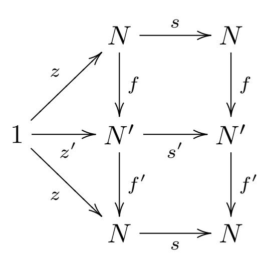
(ii) *two arrows $1 \xrightarrow{z} N \xrightarrow{s} N$*

such that

> $1 \xrightarrow{z} N \xrightarrow{s} N$ *satisfies the equivalent Lawvere, Dedekind, or Freyd axioms for a natural number object.*

Proposition 12.5.6 *In any topos, natural number objects are unique up to isomorphism.*

Proof. This is most easily seen using Lawvere's axiom. If $1 \xrightarrow{z} N \xrightarrow{s} N$ and $1 \xrightarrow{z'} N' \xrightarrow{s'} N'$ are both natural number objects, then there are unique arrows $N \xrightarrow{f} N'$ and $N' \xrightarrow{f'} N$ such that the following diagram commutes:



Since Id_N also makes the outer diagram commute, we have $f'f = \mathsf{Id}_N$. By a similar argument, $ff' = \mathsf{Id}_{N'}$. □

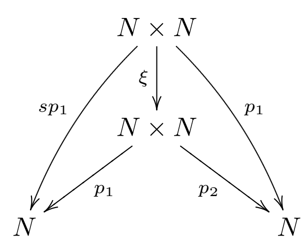
We now prove (1) ⇒ (2) from Theorem 12.5.4: that is, we show that every $1 \xrightarrow{z} N \xrightarrow{s} N$ that satisfies the Lawvere axioms for a natural number object also satisfies the Dedekind axioms.

First, by a clever construction, we define the predecessor function for $1 \xrightarrow{z} N \xrightarrow{s} N$. That is,

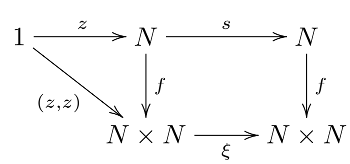
Proposition 12.5.7 *If $1 \xrightarrow{z} N \xrightarrow{s} N$ is a natural number object, then there is an arrow $N \xrightarrow{p} N$ such that*

$$\begin{aligned} pz &= z \\ ps &= \mathsf{Id}_N \end{aligned}$$

Proof. We begin by letting $N \times N \xrightarrow{\xi} N \times N$ be the unique arrow that makes the following diagram commute:



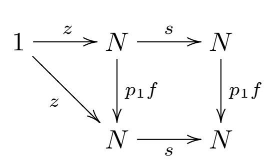
Then, since $1 \xrightarrow{z} N \xrightarrow{s} N$ satisfies the Lawvere axioms for a natural number object, there is a unique arrow $N \xrightarrow{f} N \times N$ such that the following diagram commutes:



We define $N \xrightarrow{p} N$ as follows: let $p = p_2 f$. Clearly

$$pz = p_2 fz = p_2(z, z) = z$$

Also, by the left-hand half of the product diagram above, $sp_1 = p_1\xi$, so $sp_1 f = p_1\xi f = p_1 fs$. Thus, the following diagram commutes:



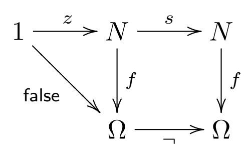
Thus, $p_1 f = \mathsf{Id}_N$. But, by right-hand side of the product diagram above, $p_1 = p_2\xi$, so $\mathsf{Id}_N = p_1 f = p_2 \xi f = p_2 f s$. Thus, $p = p_2 f$ is a left-inverse for s, as required. □

Now we can prove (1) ⇒ (2) from above:

Proposition 12.5.8 *In a non-degenerate topos, if* $1 \xrightarrow{z} N \xrightarrow{s} N$ *satisfies the Lawvere axiom for a natural number object, then it satisfies the Dedekind axioms.*

Proof. Suppose $1 \xrightarrow{z} N \xrightarrow{s} N$ satisfies the Lawvere axiom.

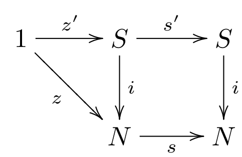
(1) If $sx = z$, then $x = psx = pz = z$. Now suppose $sz = z$ and consider $1 \xrightarrow{\mathsf{false}} \Omega \xrightarrow{\neg} \Omega$, where $\neg \circ \mathsf{false} = \mathsf{true}$. Then there is $N \xrightarrow{f} \Omega$ such that



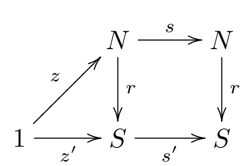
So $fz = \mathsf{false}$ and $fsz = \mathsf{true}$. But, since $sz = z$, it follows that $\mathsf{true} = fsz = fz = \mathsf{false}$. Thus $\mathsf{true} = \mathsf{false}$ and thus, by Proposition 12.1.2, $\mathcal{E}$ is degenerate.

(2) By Proposition 12.5.7, p is a left-inverse for s. Thus, s is monic.

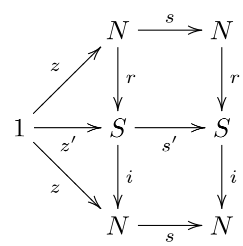
(3) Suppose $S \overset{i}{\rightarrowtail} N$ and suppose that there are arrows $1 \xrightarrow{z'} S \xrightarrow{s'} S$ such that the following diagram commutes:



Then, by the definition of N, there is a unique $N \xrightarrow{r} S$ such that the following diagram commutes:



Combining these diagrams (placing the second on top of the first), we get the following diagram, which commutes:



Thus, $i \circ r$ is an arrow from N to N that satisfies the following recursion equations:

$$\begin{array}{rcl} irz & = & z \\ sir & = & irs \end{array}$$

But Id_N also satisfies these recursion equations. Thus, $ir = \mathsf{Id}_N$. Thus, i is right-invertible. And, in any topos, a right-invertible monic is an iso. Thus, $S \overset{i}{\cong} N$, as required.

This completes our proof. □

12.6 Toposes that satisfy Choice

Definition 12.6.1 *Given a topos $\mathcal{E}$, we say that* Choice holds in $\mathcal{E}$ *if every epic is a split epic; that is, every right-cancellable arrow is right-invertible.*

Proposition 12.6.2 *$\mathcal{E}$ is well-pointed if, and only if, $\mathcal{E}$ is Boolean, bivalent, and satisfies Choice.*

12.7 The category of abstract sets and arbitrary mappings

In *Sets for Mathematics* ([**?**]), Lawvere and Rosebrugh characterize the category of abstract sets and arbitrary mappings as a non-degenerate, well-pointed topos that contains a natural number object and in which Choice holds.

Chapter 13

Functors

So far, we have introduced certain properties of a category—such as the property of containing products, pullbacks, exponentials, subobject classifiers, etc.—by appealing to the intrinsic features of that category. But sometimes we wish to introduce properties by appealing to extrinsic features. And to do that, we need the notion of a functor.

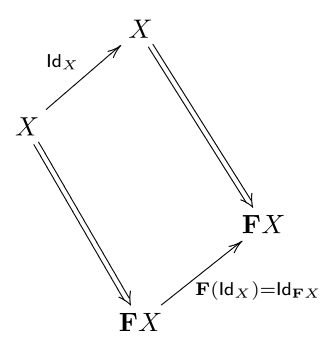
13.1 The definition of a functor

Definition 13.1.1 (Functor) *If $\mathcal{X}$ and $\mathcal{A}$ are categories, a* functor from $\mathcal{X}$ to $\mathcal{A}$ *is a function* $\mathbf{F}$ *on the collection of all $\mathcal{X}$-objects and $\mathcal{X}$-arrows such that:*

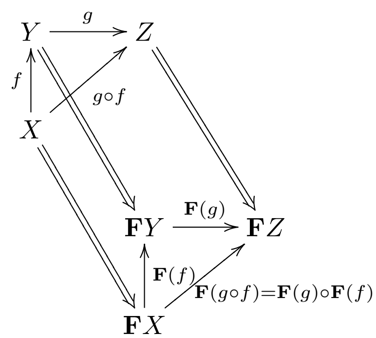
(1) $\mathbf{F}$ *takes an $\mathcal{X}$-object X to an $\mathcal{A}$-object $\mathbf{F}X$.*

(2) $\mathbf{F}$ *takes an $\mathcal{X}$-arrow $X \xrightarrow{f} Y$ to an $\mathcal{A}$-arrow $\mathbf{F}X \xrightarrow{\mathbf{F}f} \mathbf{F}Y$ such that*

  (a) $\mathbf{F}(\mathsf{Id}_X) = \mathsf{Id}_{\mathbf{F}X}$



(b) $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$



Thus, as Mac Lane urges, we might think of a functor as producing a 'picture' of $\mathcal{X}$ in $\mathcal{A}$. This way of thinking is very much encouraged by the diagrams illustrating conditions (2)(a) and (2)(b) in the above definition. Though beware: nothing in the definition of a functor rules out the case in which X, Y, and Z are all distinct, yet $\mathbf{F}X = \mathbf{F}Y = \mathbf{F}Z$.

Proposition 13.1.2 *Functors compose.*

That is, if $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ *and* $\mathbf{G} : \mathcal{A} \Rightarrow \mathcal{Y}$ *are functors, then* $\mathbf{G} \circ \mathbf{F} : \mathcal{X} \Rightarrow \mathcal{Y}$ *is a functor.*

Just as there are different sorts of arrows in a category—epis, monics, isos—there are different sorts of functors between categories.

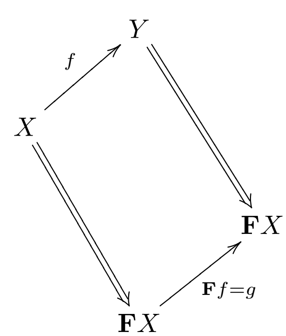
Definition 13.1.3 (Isomorphism, full, faithful) *Suppose* $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ *is a functor. Then*

(1) $\mathbf{F}$ is an isomorphism *if*

- (a) $\mathbf{F}$ *is a bijection between objects*
- (b) $\mathbf{F}$ *is a bijection between arrows*

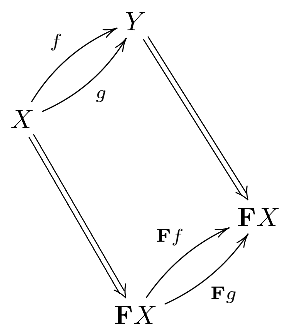
(2) $\mathbf{F}$ is full *if*

> *For any* $\mathcal{X}$*-objects* X *and* Y *and any* $\mathcal{A}$*-arrow* $\mathbf{F}X \xrightarrow{g} \mathbf{F}Y$*, there is an* $\mathcal{X}$*-arrow* $X \xrightarrow{f} Y$ *such that* $\mathbf{F}f = g$*.*



(3) **F** is faithful *if*

$$\textit{For any } X \overset{f \neq g}{\rightrightarrows} Y,\ \mathbf{F}X \overset{\mathbf{F}f \neq \mathbf{F}g}{\rightrightarrows} \mathbf{F}Y.$$



Beware: *This imposes a condition of injectivity on the function* **F** *from* arrows *to* arrows*; the corresponding map* **F** *from* objects *to* objects *may, nonetheless, not be injective.*

13.2 Examples of functors

The most basic functor is the identity functor from a category to itself:

Example 16 *Given any category* $\mathcal{C}$*, define the* identity functor $\mathbf{Id}_{\mathcal{C}} : \mathcal{C} \Rightarrow \mathcal{C}$ *as follows:*

(a) $\mathbf{Id}_{\mathcal{C}} X = X$

(b) $\mathbf{Id}_{\mathcal{C}} f = f$

The next example is the 'underlying set' functor from Grp to Set. We will meet this again in chapter 15.

Example 17 *Define the following functor* $\mathbf{U} : \mathsf{Grp} \Rightarrow \mathsf{Set}$*:*

(a) $\mathbf{U}(G, *_G) = G$

(b) $\mathbf{U}((G, *_G) \xrightarrow{h} (H, *_H)) = G \xrightarrow{h} H$

Proposition 13.2.1 **U** *is not full, but it is faithful.*

Proof. **U** *is not full.* If $(G, *_G)$ is not the trivial group, there is $g \in G$ such that $g \neq e_G$. Then there are set functions $f : G \to G$ such that $f(e_G) = g$. There is no group homomorphism h such that $\mathbf{U}h = f$.

U *is faithful.* If $(G, *_G) \overset{h_1 \neq h_2}{\rightrightarrows} (H, *_H)$ are distinct group homomorphisms, then they must differ on at least one element of G. Thus, as set functions they are distinct. □

Inspired by this example, we introduce an intuitive notion. A functor from a category of richer structures to a category of less rich structures that extracts the less rich structure from the richer structure is called a *forgetful functor*. Thus, the functor $\mathbf{U} : \mathsf{Grp} \Rightarrow \mathsf{Set}$ is a forgetful functor because it extracts the less rich structure, namely, the set structure.

Example 18 *Let* Ab *be the category of Abelian groups and their group homomorphisms and let* Ring *be the category of rings and ring homomorphisms.*

Define the following functor $\mathbf{W} : \mathsf{Ring} \Rightarrow \mathsf{Ab}$*:*

(a) $\mathbf{W}(R, +, \times) = (R, +)$

(b) $\mathbf{W}((R, +, \times) \stackrel{h}{\to} (R', +', \times')) = (R, +) \stackrel{h}{\to} (R', +')$

Proposition 13.2.2 *Then* $\mathbf{W}$ *is not full, but it is faithful.*

Proof. The reasons are similar to those above, though it is more difficult to give an example of a group homomorphism between the two underlying Abelian groups of two rings to which $\mathbf{W}$ does not map. However, consider the ring of integers $(\mathbb{Z}, +, \times)$ and the function $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = -n$. Then h is a group endomorphism on $(\mathbb{Z}, +)$, but it is not a ring endomorphism on $(\mathbb{Z}, +, \times)$. □

Again, $\mathbf{W}$ is a forgetful functor. It extracts the (Abelian) group structure of the ring.

Example 19 *Define* $\mathbf{P}_1, \mathbf{P}_2 : \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C}$*:*

(a) (i) $\mathbf{P}_1((X, Y)) = X$
    (ii) $\mathbf{P}_2((X, Y)) = Y$

(b) (i) $\mathbf{P}_1((X_1, Y_1) \stackrel{(f_1, f_2)}{\to} (X_2, Y_2)) = X_1 \stackrel{f_1}{\to} X_2$
    (ii) $\mathbf{P}_2((X_1, Y_1) \stackrel{(f_1, f_2)}{\to} (X_2, Y_2)) = Y_1 \stackrel{f_2}{\to} Y_2$.

Proposition 13.2.3 $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are full, but not faithful.*

Example 20 *Given a category* $\mathcal{C}$*, define* $\Delta : \mathcal{C} \Rightarrow \mathcal{C} \times \mathcal{C}$*:*

(a) $\Delta(A) = (A, A)$

(b) $\Delta(A \stackrel{f}{\to} B) = (A, A) \stackrel{(f, f)}{\to} (B, B)$

This functor is called the diagonal functor.

Proposition 13.2.4 Δ *is not full, but it is faithful.*

Example 21 *If* $\mathcal{C}$ *contains products and coproducts, define the following functors* $+, \times : \mathcal{C} \times \mathcal{C} \Rightarrow \mathcal{C}$*:*

(a) (i) $+(X, Y) = X + Y$

(ii) $\times(X, Y) = X \times Y$

(b) (i) $+((X_1, Y_1) \stackrel{(f_1, f_2)}{\rightarrow} (X_2, Y_2)) = f_1 + f_2$

(ii) $\times((X_1, Y_1) \stackrel{(f_1, f_2)}{\rightarrow} (X_2, Y_2)) = f_1 \times f_2$

Example 22 *If $\mathcal{C}$ contains exponentials, then given a $\mathcal{C}$-object X, define the functor $(-)^X : \mathcal{C} \Rightarrow \mathcal{C}$:*

(a) $(-)^X(Y) = Y^X$

(b) $(-)^X(Y_1 \stackrel{f}{\rightarrow} Y_2) = Y_1^X \stackrel{f \circ \_}{\rightarrow} Y_2^X$, *where* $(f \circ \_)(h : X \rightarrow Y_1) = f \circ h$.

We will meet these last two examples again in chapter 15.

Chapter 14

Natural transformations

When Mac Lane and Eilenberg first introduced the notion of a category, they did so in order to introduce the notion of functor, and they introduced the notion of functor in order to introduce the notion of a natural transformation between two functors. We define this final notion here and give some examples.

Definition 14.0.5 (Natural transformation) *Given two functors* $\mathbf{F}, \mathbf{G} : \mathcal{X} \Rightarrow \mathcal{A}$*, a* natural transformation from $\mathbf{F}$ to $\mathbf{G}$ *consists of*

(1) *a family of* $\mathcal{A}$*-arrows*

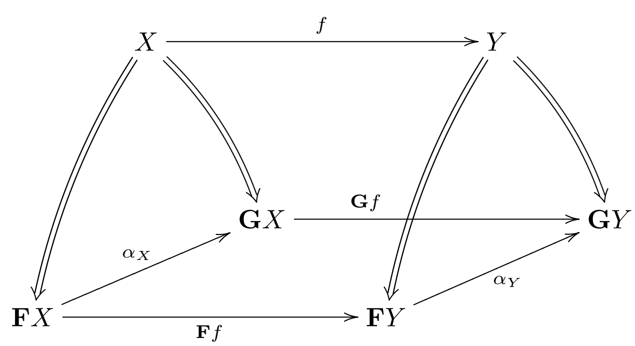
$$\alpha = \{\alpha_X : X \textit{ is an } \mathcal{X}\textit{-object}\}$$

where

$$\mathbf{F}X \xrightarrow{\alpha_X} \mathbf{G}X$$

such that

> *For any* $\mathcal{X}$*-objects* X *and* Y *and any* $\mathcal{X}$*-arrow* $X \xrightarrow{f} Y$*, the square on the base of the following diagram commutes:*



If each α_X *is an iso, we say that the natural transformation* α *is a* natural equivalence *or* natural isomorphism*.*

Thus, if we think of a functor $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ as producing a picture of $\mathcal{X}$ in $\mathcal{A}$, we might think of a natural transformation as a transformation of one picture into another.

Proposition 14.0.6 *Natural transformations compose.*

Suppose $\mathbf{F}, \mathbf{G}, \mathbf{H} : \mathcal{X} \Rightarrow \mathcal{A}$ *are functors. Then, if* $\eta : \mathbf{F} \to \mathbf{G}$ *and* $\varepsilon : \mathbf{G} \to \mathbf{H}$ *are natural transformations then* $\varepsilon \circ \eta : \mathbf{F} \to \mathbf{H}$ *is a natural transformation, where, if* X *is in* $\mathcal{X}$*,*

$$(\varepsilon \circ \eta)_X = \varepsilon_X \circ \eta_X$$

We begin with a simple and rather uninteresting example, simply to illustrate the definition:

Example 23 *Define the following functor* $\mathbf{P} : \mathsf{Set} \Rightarrow \mathsf{Set}$*:*

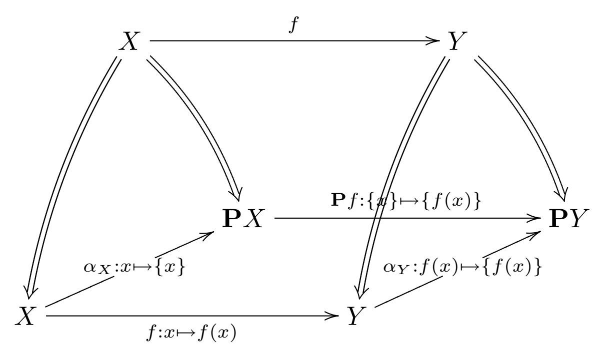
(a) $\mathbf{P}X =$ *the power set of* X

(b) $\mathbf{P}f(S) =$ *the image of* S *under* f

Then, for each X *in* Set*, let* $X \xrightarrow{\alpha_X} \mathbf{P}X$ *be the following arrow in* Set*:* $\alpha_X : x \mapsto \{x\}$*.*

Then $\alpha : \mathbf{Id}_{\mathsf{Set}} \to \mathbf{P}$ *is a natural transformation.*

Proof. The image of $\{x\}$ under f is $\{f(x)\}$, which is also the singleton of $f(x)$.



□

Staying with Set, the next two examples are a little more interesting.

Example 24 *Given a set* X*, define the* collapse function $X + X \xrightarrow{c_X} X$ *as follows:* $c_X : x \mapsto x$*.*

Then $c = \{c_X : X \text{ is a set}\}$ *is a natural transformation* $c : + \circ \Delta \to \mathbf{Id}_{\mathsf{Set}}$*.*

Example 25 *Given any two sets* X *and* Y*, let* $X \xrightarrow{i_X} X + Y$ *be the inclusion function that takes* X *to itself and let* $Y \xrightarrow{i_Y} X + Y$ *be the inclusion function* Y *that takes* Y *to itself.*

Then $i = \{(i_X, i_Y) : (X, Y) \text{ in } \mathsf{Set} \times \mathsf{Set}\}$ *is a natural transformation* $i : \mathbf{Id}_{\mathsf{Set} \times \mathsf{Set}} \to \Delta \circ +$.

In the next chapter, we will be particularly interested in functors $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ and $\mathbf{G} : \mathcal{A} \to \mathcal{X}$ such that there are natural transformations:

$$\alpha : \mathbf{Id}_{\mathcal{X}} \to \mathbf{GF} \quad \text{and} \quad \beta : \mathbf{FG} \to \mathbf{Id}_{\mathcal{A}}$$

that satisfy certain further naturality conditions. They will be called *adjoint functors*. First, however, we will see some examples of natural transformations for areas of mathematics outside elementary set theory.

Example 26 *Let* CRng *be the category of all commutative rings and ring homomorphisms between them.*

Define the following two functors $\mathbf{M}, \mathbf{Y} : \mathsf{CRng} \Rightarrow \mathsf{Grp}$*:*

(a) $\mathbf{M}K = \mathrm{GL}_n K$*, the group of* $n \times n$ *matrices over* K

(b) $\mathbf{M}f =$ *the natural group homomorphism induced by the ring homomorphism* f *on the underlying commutative ring.*

and

(a') $\mathbf{Y}K = K^*$*, where* K^* *is the group of invertible elements of* K*.*

(b') $\mathbf{Y}f = f|_{K^*}$.

Then, for each K *in* CRng*, let* $\det_K : \mathbf{M}(K) = \mathrm{GL}_n K \to K^* = \mathbf{Y}(K)$ *be the determinant function.*

Then $\det_K : \mathbf{M} \to \mathbf{Y}$ *is a natural transformation.*

Example 27 *Define the following functor* $(-)^{\mathrm{op}} : \mathsf{Grp} \Rightarrow \mathsf{Grp}$*:*

(a) $(G, *_G)^{\mathrm{op}} = (G, *^{\mathrm{op}})$*, where* $a *^{\mathrm{op}} b = b *_G a$*.*

(b) *If* $(G, *_G) \xrightarrow{\varphi} (H, *_H)$*, then* $\varphi^{\mathrm{op}}(a *^{\mathrm{op}} b) = \varphi(b *_G a)$*.*

Then, for each G*, define* $G \xrightarrow{\alpha_G} G^{\mathrm{op}}$ *as follows:* $\alpha_G : g \mapsto g^{-1}$*.*

Then $\alpha : \mathbf{Id}_{\mathsf{Grp}} \to (-)^{\mathrm{op}}$ *is a natural transformation since* $\varphi^{\mathrm{op}}(g^{-1}) = (\varphi^{\mathrm{op}}(g))^{-1}$ *for group homomorphism* $G \xrightarrow{\varphi} H$*.*

Chapter 15

Adjoint functors

15.1 Definition

Above, we saw that there are natural transformations

$$c : + \circ \Delta \to \mathbf{Id}_{\mathsf{Set}} \text{ and } i : \mathbf{Id}_{\mathsf{Set}\times\mathsf{Set}} \to \Delta \circ +$$

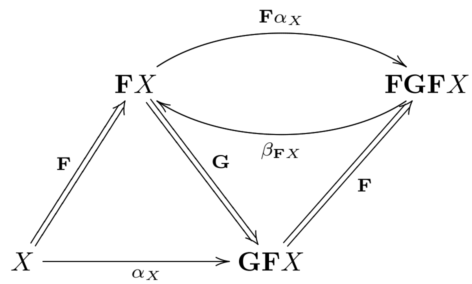
Functors for which there are such natural transformations are called *adjoint functors* if they satisfy a certain further condition. We state this here:

Definition 15.1.1 (Adjoint functors) *Given functors* $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ *and* $\mathbf{G} : \mathcal{A} \Rightarrow \mathcal{X}$*, we say that* $\mathbf{F}$ is left adjoint to $\mathbf{G}$ *or, equivalently,* $\mathbf{G}$ is right adjoint to $\mathbf{F}$ (*written* $F \dashv G$) *if*

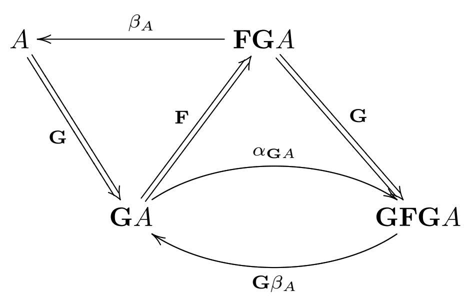
> *There exist natural transformations*
>
> $$\alpha : \mathbf{Id}_{\mathcal{X}} \to \mathbf{GF} \text{ and } \beta : \mathbf{FG} \to \mathbf{Id}_{\mathcal{A}}$$
>
> *such that*
>
> (a) *For each object* X *of* $\mathcal{X}$
>
> $$\beta_{\mathbf{F}X} \circ \mathbf{F}\alpha_X = \mathsf{Id}_{\mathbf{F}X}$$
>
> (b) *For each object* A *of* $\mathcal{A}$
>
> $$\mathbf{G}\beta_A \circ \alpha_{\mathbf{G}A} = \mathsf{Id}_{\mathbf{G}A}$$

α *is called the* unit *and* β *is called the* counit.

Condition (a) demands that, for any X in $\mathcal{X}$, the two $\mathcal{A}$-arrows in the following *zig-zag diagram* compose to give $\mathsf{Id}_{\mathbf{F}X}$:



Condition (b) demands that, for any A in $\mathcal{A}$, the two $\mathcal{X}$-arrows in the following *zig-zag diagram* compose to give $\mathsf{Id}_{\mathbf{G}A}$:



15.2 Properties of adjoints

Proposition 15.2.1 *Left and right adjoints are unique up to isomorphism.*

That is, suppose $\mathbf{F}, \mathbf{F}' : \mathcal{X} \Rightarrow \mathcal{A}$ *and* $\mathbf{G}, \mathbf{G}' : \mathcal{A} \Rightarrow \mathcal{X}$ *are functors. Then*

(a) *If* $\mathbf{F} \dashv \mathbf{G}$ *and* $\mathbf{F}' \dashv \mathbf{G}$, *then there is a natural isomorphism* $\alpha : \mathbf{F} \to \mathbf{F}'$.

(b) *If* $\mathbf{F} \dashv \mathbf{G}$ *and* $\mathbf{F} \dashv \mathbf{G}'$, *then there is a natural isomorphism* $\beta : \mathbf{G} \to \mathbf{G}'$.

Proposition 15.2.2 *Any functor naturally isomorphic to a left (or right) adjoint is itself a left (or right) adjoint.*

Proposition 15.2.3 *Adjoints compose.*

That is, suppose $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$, $\mathbf{G} : \mathcal{A} \Rightarrow \mathcal{X}$, $\mathbf{F}' : \mathcal{A} \Rightarrow \mathcal{B}$, $\mathbf{G}' : \mathcal{B} \Rightarrow \mathcal{A}$ *are functors and suppose*

$$\mathbf{F} \dashv \mathbf{G} \text{ and } \mathbf{F}' \dashv \mathbf{G}'$$

Then

$$\mathbf{F}' \circ \mathbf{F} \dashv \mathbf{G} \circ \mathbf{G}'$$

The most important property of adjoints is that they preserve 'set-sized' or 'small' diagrams:

Proposition 15.2.4

(a) *Right adjoints preserve limits of small diagrams.*

(b) *Left adjoints preserve colimits of small diagrams.*

That is, suppose $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ *and* $\mathbf{G} : \mathcal{A} \Rightarrow \mathcal{X}$. *And suppose* $\mathbf{F} \dashv \mathbf{G}$. *Then*

(a) *If* $\mathbf{D}$ *is a small diagram in* $\mathcal{A}$ *and*

$$(Q, \{Q \stackrel{q_D}{\rightarrow} D : D \textit{ in } \mathbf{D}\})$$

is a limit for $\mathbf{D}$ *in* $\mathcal{A}$*, then*

$$(\mathbf{G}Q, \{\mathbf{G}Q \stackrel{\mathbf{G}q_D}{\rightarrow} \mathbf{G}D : D \textit{ in } \mathbf{D}\})$$

is a limit for $\mathbf{G}(\mathbf{D})$ *in* $\mathcal{X}$.

(b) *If* $\mathbf{D'}$ *is a small diagram in* $\mathcal{X}$ *and*

$$(Q', \{D' \stackrel{q_{D'}}{\rightarrow} Q' : D' \textit{ in } \mathbf{D'}\})$$

is a colimit for $\mathbf{D'}$ *in* $\mathcal{X}$*, then*

$$(\mathbf{F}Q', \{\mathbf{F}D' \stackrel{\mathbf{F}q_{D'}}{\rightarrow} \mathbf{F}Q' : D' \textit{ in } \mathbf{D'}\})$$

is a colimit for $\mathbf{F}(\mathbf{D'})$ *in* $\mathcal{A}$.

15.3 Examples concerning Set

Proposition 15.3.1

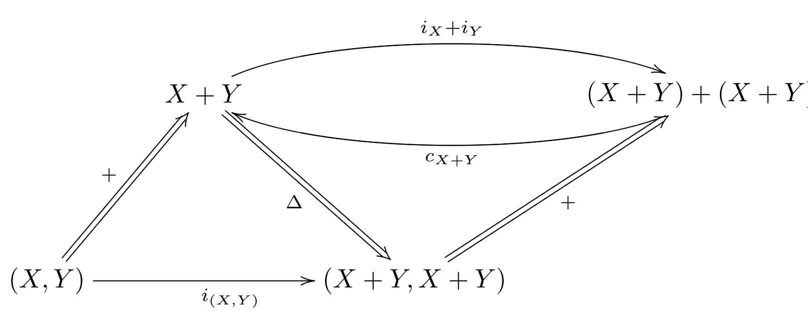
$$+ \dashv \Delta$$

Proof. From above, we have natural transformations

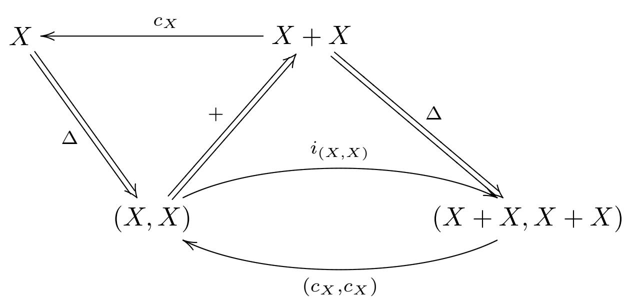
$$i : \mathbf{Id}_{\mathsf{Set}\times\mathsf{Set}} \rightarrow \Delta \circ + \quad \text{and} \quad c : + \circ \Delta \rightarrow \mathbf{Id}_{\mathsf{Set}}$$

Thus, all that remains is to observe that c and i satisfy the following zig-zag diagrams:

(a) Suppose (X, Y) is in $\mathsf{Set} \times \mathsf{Set}$:



(b) Suppose X is in Set:



This completes our proof. $\square$

Proposition 15.3.2

$$\Delta \dashv \times$$

Proof. Given X in Set, define

$$X \overset{\delta_X}{\to} X \times X$$

as follows: $\delta_X : x \mapsto (x,x)$. Then $\delta : \mathbf{Id}_{\mathsf{Set}} \to \times \circ \Delta$ is a natural transformation.

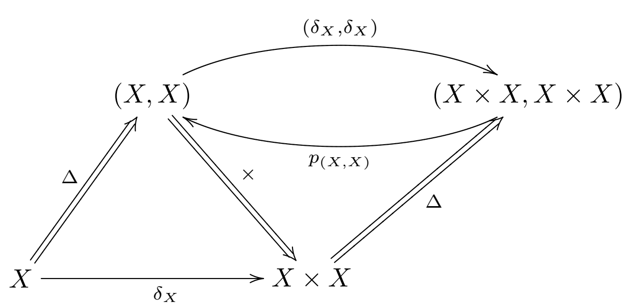
Given $(X \times Y, X \times Y)$ in $\mathsf{Set} \times \mathsf{Set}$, define

$$(X \times Y, X \times Y) \overset{p_{(X,Y)}}{\to} (X,Y)$$

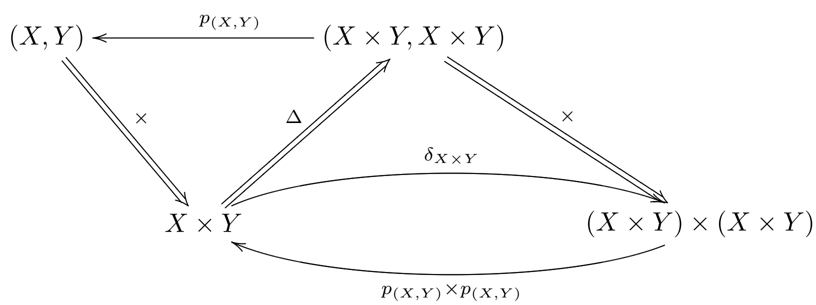
as follows: $p_{(X,Y)} = (p_1, p_2)$, where $p_1 : (x,y) \mapsto x$ and $p_2 : (x,y) \mapsto y$. Then $\beta : \Delta \circ \times \to \mathbf{Id}_{\mathsf{Set}\times\mathsf{Set}}$ is a natural transformation.

Now all that remains is to observe that δ and p satisfy the following zig-zag diagrams:

(a) Suppose X is in Set:



(b) Suppose (X, Y) is in $\mathsf{Set} \times \mathsf{Set}$:



This completes our proof. □

Proposition 15.3.3

$$(- \times X) \dashv (-)^X$$

Proof. Given A in Set, define

$$A \overset{k_A}{\to} (A \times X)^X$$

as follows: $k_A : a \mapsto (x \mapsto (a, x))$. Then $k : \mathbf{Id}_{\mathsf{Set}} \to (-)^X \circ (- \times X)$ is a natural transformation.

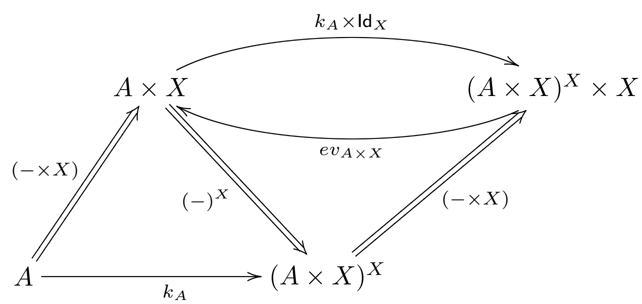
Given A in Set, let

$$A^X \times X \overset{ev_A}{\to} A$$

be the evaluation map. Then $ev : (- \times X) \circ (-)^X \to \mathbf{Id}_{\mathsf{Set}}$ is a natural transformation.

Now all that remains is to show that k and ev satisfy the following zig-zag diagrams:

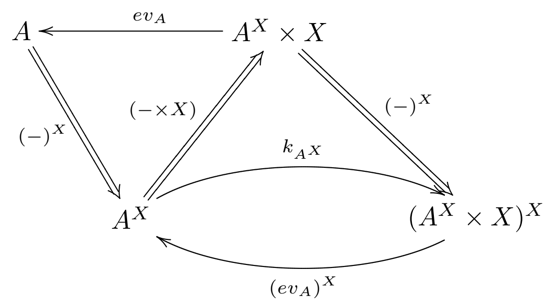
(a) Suppose A is in Set:



Then $ev_{A \times X} \circ (k_A \times \mathsf{Id}_X) = \mathsf{Id}_{A \times X}$ since

$$(a, x) \overset{k_A \times \mathsf{Id}_A}{\mapsto} (x \mapsto (a, x), x) \overset{ev_{A \times X}}{\mapsto} (a, x)$$

(b) Suppose A is in Set:



Then $(ev_A)^X \circ k_{A^X} = \mathsf{Id}_{A^X}$ since

$$(x \mapsto f(x)) \overset{k_{A^X}}{\mapsto} (x \mapsto (f, x)) \overset{(ev_A)^X}{\mapsto} (x \mapsto (f, x) \overset{ev_A}{\mapsto} f(x))$$

This completes our proof. □

15.4 Examples concerning forgetful functors

Recall our brief discussion of forgetful functors from chapter 13. A forgetful functor takes a category with richer structures—such as Grp—to a category with less rich structures—such as Set—in such a way that it extracts the less rich structure—for instance, the set structure—from the richer structure—the group structure. As we will see in this section, for many forgetful functors, their left adjoint is the functor that takes a less rich structure—for instance, a set—and constructs a richer structure—a group—in the natural way, where the natural way will often turn out to involve taking the free structure of the less rich structure—for instance, the free group over a particular set.

Definition 15.4.1 (Underlying set functor) *Recall that the* underlying set functor $\mathbf{U} : \mathsf{Grp} \Rightarrow \mathsf{Set}$ *is the forgetful functor that takes* $(G, *)$ *to its underlying set* G.

Definition 15.4.2 (Free group functor) *Define the* free group functor $\mathbf{E} : \mathsf{Set} \Rightarrow \mathsf{Grp}$ *as follows:*

(a) $\mathbf{E}X$ = *the free group on* X*—that is, the underlying set of* $\mathbf{E}X$ *is the set of 'words' or sequences*

$$(x_1, ..., x_n)$$

where $x_1, x_2, ..., x_n \in X$ *and the binary operation of* $\mathbf{E}X$ *is*

$$(x_1, ..., x_n) * (x'_1, ..., x'_m) = (x_1, ...x_n, x'_1, ..., x'_m)$$

(b) $\mathbf{E}f : (x_1, ..., x_n) \mapsto (f(x_1), ..., f(x_n))$, *whenever* $f : X \to Y$.

Proposition 15.4.3

$$\mathbf{E} \dashv \mathbf{U}$$

Proof. Given X in Set, define

$$X \xrightarrow{i_X} \mathbf{UE}X$$

as follows: $i_X : x \mapsto (x)$. Then $i : \mathbf{Id}_{\mathsf{Set}} \to \mathbf{UE}$ is a natural transformation.

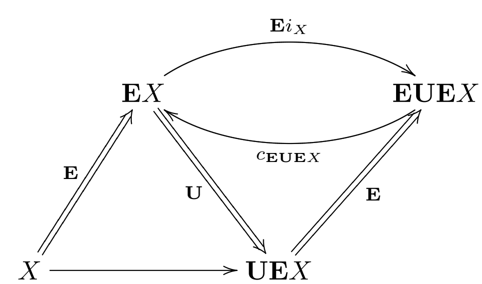
Given G in Grp, define

$$\mathbf{EU}G \xrightarrow{c_G} G$$

as follows: $c_G : (g_1, ..., g_n) \mapsto g_1 *_G ... *_G g_n$. Then $c : \mathbf{EU} \to \mathbf{Id}_{\mathsf{Grp}}$ is a natural transformation.

Now all that remains is to show that i and c satisfy the following zig-zag diagrams:

(a) Suppose X is in Set:



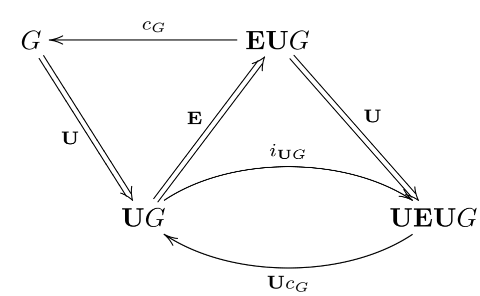
Then $c_{\mathbf{EUE}X} \circ \mathbf{E}i_X = \mathsf{Id}_{\mathbf{E}X}$ since

$$(x_1, ..., x_n) \overset{i_{\mathbf{E}X}}{\mapsto} ((x_1, ..., x_n)) \overset{c_{\mathbf{EUE}X}}{\mapsto} (x_1, ..., x_n)$$

After all

$$((x_1, ..., x_n), ..., (x'_1, ..., x'_m)) \overset{c_{\mathbf{EUE}X}}{\mapsto} (x_1, ..., x_n, ..., x'_1, ..., x'_m)$$

(b) Suppose G is in Grp:



Then $\mathbf{U}c_G \circ i_{\mathbf{U}G} = \mathsf{Id}_{\mathbf{U}G}$ since

$$g \overset{i_{\mathbf{U}G}}{\mapsto} (g) \overset{\mathbf{U}c_G}{\mapsto} g$$

After all

$$(g_1, ..., g_n) \overset{\mathbf{U}c_G}{\mapsto} g_1 *_G ... *_G g_n$$

This completes our proof. □

More generally, for many forgetful functors $\mathbf{G} : \mathcal{X} \Rightarrow \mathcal{A}$, there is a functor $\mathbf{F} : \mathcal{X} \Rightarrow \mathcal{A}$ that constructs a richer structure $\mathbf{F}X$ in $\mathcal{A}$ from the less rich structure X in $\mathcal{X}$ such that $\mathbf{F} \dashv \mathbf{G}$. For instance,

- The construction of the free vector space over a set is left adjoint to the underlying set functor from Vect_K to Set.
- The factor-commutator construction of an Abelian group $G/[G, G]$ from G is left adjoint to the forgetful functor that inserts Ab into Grp.
- The construction of a 'field of fractions' over an integral domain is left adjoint to the forgetful functor that inserts Field, the category of fields, into Dom, the category of intergral domains.
- The Stone-Čech compactification of a topological space is left adjoint to the forgetful functor from KHaus, the category of compact Hausdorff spaces to Top, the category of topological spaces.

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